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LOOPS, THEIR CORES AND SYMMETRIC SPACES

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ABSTRACT

This paper is devoted to the relations among affine symmetric spaces, smooth Bol and Moufang loops, smooth left distributive quasigroups and differentiable 3-nets. The results are used to prove the analyticity of smooth Moufang loops and left distributive quasigroups with involutive left translations as well as to show the Lie nature of transformation groups naturally related to some classes of smooth binary systems and 3-nets. In the last section we establish power series expansion for local loops with weak associativity conditions and apply the methods of the previous sections in order to describe geodesic loops having euclidean lines either as their geodesic lines or as geodesic lines of their core.

§0. Introduction

Although the theory of symmetric spaces has been used for the investigation of local analytical Moufang and Bol loops ([20], [21] pp. 381–398, [1]), the relations between global symmetric spaces, global smooth Bol and Moufang loops and differentiable global 3-nets are until now not sufficiently explored. The main body of this paper intends to fill this gap. We use the consequences of these

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relations for local smooth loops to establish power series expansion for loops with weak associativity conditions and to classify special classes of alternative local Lie loops.

At the beginning of the first section and in Theorem 2.9 we treat groups naturally related to differentiable connected Bol and Moufang loops L by means of the associated 3-nets. In particular, we investigate the topological projectivity groups, the groups topologically generated by the left, the right and by all translations of L, the groups of continuous automorphisms, semi-automorphisms and pseudo-automorphisms. The results are either generalizations of the statements given in [2] or yield alternative proofs.

Motivated by the ideas of [1] we introduce on the horizontal and transversal lines of the 3-net associated with a differentiable Bol loop a symmetric space structure in the sense of O. Loos [18] using certain projectivities which are induced by involutory collineations, called Bol reflections. The methods developed in these investigations allow us to prove that the group generated by the Bol reflections as well as the group (topologically) generated by the left translations of a Bol loop are Lie groups. For the second result there is in the case of local Bol loops an alternative proof given by Miheev and Sabinin [21], p. 424. For differentiable Bol loops which are not Moufang loops the group topologically generated by the right translations usually fails to be a Lie group because of its infinite dimension.

A main result of section 1 is the fact (which we prove geometrically) that every differentiable connected Moufang loop is analytic. This permits us to use for differentiable Moufang loops the highly developed machinery of analytical Moufang loops which has the same level as the theory of Lie groups; in particular all differentiable simple Moufang loops are classified by the simple Malcev algebras. For differentiable Bol loops L we are able to show partial results in the same direction: if the multiplication of L is analytic resp. of the class C^r then L is analytic resp. of the class C^r (which means that the other two binary operations associated with L are in the same category). At the end of the first section we prove that every connected differentiable Moufang loop is a G-loop.

In the second section we deepen the relations between Bol loops and their cores which are left distributive quasigroupoids and which have been studied by Belousov [3]. If the core of a Bol loop is a quasigroup then it is isotopic to a Bol loop which is a left A-loop and has the automorphic inverse property. The cores of isotopic Bol loops are isomorphic. The main result of the second section concerns left distributive quasigroups. We prove that any differentiable connected left distributive quasigroup the left translations of which are involutions is analytic and that the group topologically generated by the left translations is a Lie group.

On the one hand, the left alternative loops generalize in a natural way the left Bol loops. On the other hand, smooth left alternative local loops have local representations as geodesic loops with respect to an affine connection with vanishing curvature. Hence the multiplication of an analytic left alternative local loop can be expressed by the power series of the covariant constant vector fields of this affine connection. In this way we obtain similarly as Miheev and Sabinin in [20] generalizations of the classical Hausdorff-Campbell formula for the class of left alternative local loops. Our formulae (see (G) and (H) in §3) allow to compute recursively all coefficients of the expansion of the loop multiplication starting with coefficients of the power series of the covariant constant vector fields. For analytic left and right alternative local loops this power series expansion reduces to the classical Hausdorff-Campbell formula and shows that such local loops are diassociative. As an interesting special case we consider geodesic loops with respect to an affine connection ∇ having vanishing curvature and euclidean lines as geodesic lines and give a natural representation for them.

At the end of the paper we turn our attention to the class of differentiable local Bol loops satisfying the identity $x\cdot(y^2\cdot x) = y\cdot(x^2\cdot y)$. This class of local Bol loops is characterized by the fact that the group G (topologically) generated by the left translations is a nilpotent Lie group of class 2 which is equivalent to the property that the local core of any such loop is a commutative local group. For the loops L in this class we are able to reconstruct the tangent algebra of L in the Lie algebra of G and to express in normal coordinates explicitely the multiplication of the loop L in terms of its tangent algebra. From our representation of these loops it is clear that the class of such local loops is really rich. But none of these loops can be embedded into a global Bol loop fulfilling the identity $x \cdot (y^2 \cdot x) = y \cdot (x^2 \cdot y)$. Namely, we can prove ([27], §7) that any connected differentiable Bol loop satisfying this identity must be an abelian group.

§1. Bol and Moufang loops, their nets and symmetric spaces

A set Q with a binary operation $(x, y) \mapsto x \circ y$ is called a **quasigroup** if for any given $a, b \in Q$ the equations $a \circ y = b$ and $x \circ a = b$ have precisely one solution which we denote by $y = a\backslash b$ and $x = b/a$. The left translations λ_x: $y \mapsto x \circ y$: $Q \to Q$ and the right translations ρ_x: $y \mapsto y \circ x$: $Q \to Q$ are bijections of Q, and $(x, y) \mapsto x\backslash y = y\lambda_x^{-1}$, respectively $(x, y) \mapsto y/x = y\rho_x^{-1}$, are further binary operations on Q. If a quasigroup Q has an element 1 with

$1 \circ x = x \circ 1 = x$ then it is called a **loop** and 1 is the unit element of Q. Two quasigroups $(Q_1, \circ)$ and $(Q_2, *)$ are called **isotopic** if there are three bijections $\alpha, \beta, \gamma\colon Q_1 \to Q_2$ such that

$$\alpha(x) * \beta(y) = \gamma(x \circ y)$$

holds for any $x, y \in Q_1$.

Let $\mathcal{C}$ be the category of topological spaces, C^∞-differentiable manifolds or analytical manifolds. A quasigroup Q is a $\mathcal{C}$-quasigroup if Q is an object in the category $\mathcal{C}$ and the mappings $(x,y) \mapsto x \circ y$, $(x,y) \mapsto x\backslash y$, $(x,y) \mapsto y/x\colon Q^2 \to Q$ are $\mathcal{C}$-morphisms.

A 3-net $\mathcal{N}$ is a set of points with 3 families (pencils) of lines — the lines of these families are called horizontal, vertical and transversal lines — such that the following conditions hold:

(i) every point is incident with exactly one line of every pencil;

(ii) two lines of different families have exactly one point in common;

(iii) there exist 3 lines belonging to 3 different families which are not incident with the same point.

We denote by $\mathcal{H}$ the pencil of horizontal, by $\mathcal{V}$ the pencil of vertical and by $\mathcal{T}$ the pencil of transversal lines.

It is well known that with any loop L we can associate a 3-net $\mathcal{N}(L)$ with the families of lines:

$$\begin{aligned}
\mathcal{H} &= \big\{\{(x,y),\ x \in L\},\ y \in L\big\},\\
\mathcal{V} &= \big\{\{(x,y),\ y \in L\},\ x \in L\big\},\\
\mathcal{T} &= \big\{\{(x,y),\ x \circ y = z,\ x,y \in L\},\ z \in L\big\}.
\end{aligned}$$

Conversely, every 3-net leads to a class of isotopic loops (cf. [2], p. 8 and [3], p. 20).

A 3-net $\mathcal{N}$ is called a $\mathcal{C}$-net if the point set of $\mathcal{N}$ and any pencil of lines are objects in $\mathcal{C}$ and the following mappings

$$x \mapsto H_x\colon \mathcal{N} \to \mathcal{H}, \quad x \mapsto V_x\colon \mathcal{N} \to \mathcal{V} \quad \text{and} \quad x \mapsto T_x\colon \mathcal{N} \to \mathcal{T}$$

assigning to a point x the horizontal, vertical, respectively transversal line incident with x as well as the mappings

$$(H,V) \mapsto H \cap V\colon \mathcal{H} \times \mathcal{V} \to \mathcal{N}, \quad (T,H) \mapsto T \cap H\colon \mathcal{T} \times \mathcal{H} \to \mathcal{N},$$
$$(V,T) \mapsto V \cap T\colon \mathcal{V} \times \mathcal{T} \to \mathcal{N}$$

are morphisms of $\mathcal{C}$. The restrictions $\{G\} \times \mathcal{V} \to G$, $G \in \mathcal{H}$, $\{G\} \times \mathcal{H} \to G$, $G \in \mathcal{T}$, $\{G\} \times \mathcal{T} \to G$, $G \in \mathcal{V}$, of the last three morphisms show that any line of $\mathcal{N}$ is an object of $\mathcal{C}$.

If G_1 and G_2 are lines of the 3-net $\mathcal{N}$ and $\mathcal{D}$ is a pencil of lines not containing G_1 and G_2, then the bijection $G_1 \to G_2$ which assigns to each point $x \in G_1$ the point $D_x \cap G_2$ where D_x is the line of $\mathcal{D}$ incident with x is called the **perspectivity** along $\mathcal{D}$ from G_1 to G_2. We denote this mapping by $[G_1, \mathcal{D}, G_2]$. Compositions of perspectivities are called **projectivities** between two lines. A representation $\prod_{i=1}^{n}[G_{i-1}, \mathcal{D}_{i-1}, G_i]$ of a projectivity $\pi: G_0 \to G_n$ is called **irreducible** if $D_{i-1} \neq D_i$ $(i = 1, \ldots, n-1)$ and $G_{j-1} \neq G_j$ $(j = 1, \ldots, n)$. If the domain coincides with the range, then we obtain a projectivity of the line onto itself. The set of projectivities of a line G is called the **group of projectivities** $\Pi(G)$ of G.

Let L be a differentiable loop and $\mathcal{N}$ the associated differentiable 3-net. If L is an n-dimensional differentiable manifold then any line of $\mathcal{N}$ is a differentiable submanifold of dimension n in the $2n$-dimensional differentiable manifold $\mathcal{N}$. Any projectivity of a line G onto a line G' is a diffeomorphism and the group Π of projectivities of a line G onto itself is a transitive topological transformation group of G with respect to the Arens topology ([2], §8, [10], IX.2). The topological projectivity group $\widehat{\Pi}$ is the closure of Π in the homeomorphism group of G which also carries the Arens topology. Since for all lines the projectivity groups are isomorphic, we can associate with the 3-net $\mathcal{N}$ or with the loop L the topological transformation group $\widehat{\Pi}$.

The topological closures $\widehat{\mathcal{L}}$ and $\widehat{\mathcal{R}}$ of groups $\mathcal{L}$ and $\mathcal{R}$ generated by the left, respectively right, translations of a differentiable loop are in a natural way transformation subgroups of the topological transformation group $\widehat{\Pi}$ ([2], §6). Although for Lie groups it is evident that $\widehat{\mathcal{L}}$, $\widehat{\mathcal{R}}$ and $\widehat{\Pi}$ are Lie transformation groups, for differentiable proper loops the property of $\widehat{\mathcal{L}}$, $\widehat{\mathcal{R}}$ and $\widehat{\Pi}$ to be Lie groups depends heavily on the associativity conditions which are satisfied by the loop.

A loop L satisfying the identity $(x \cdot yx)z = x(y \cdot xz)$ respectively $z(xy \cdot x) = (zx \cdot y)x$ for all $x, y, z \in L$ is called a left respectively right **Bol loop**. In the following we use the term Bol loop for a left Bol loop. All loops isotopic to a Bol loop are also Bol loops. Any such loop has the left inverse property which means that for any element x there is an element x^{-1} with $x^{-1}x = xx^{-1} = 1$ and $x^{-1}(xy) = y$ for all $y \in L$. The loops satisfying the left and the right Bol identity are called **Moufang loops**.

Let $\mathcal{V}$ be the pencil of vertical lines and let $\mathcal{Y}_i$ $(i = 1,2)$ be the other two pencils in a 3-net $\mathcal{N}$. Let X_j $(j = 1,2)$ be lines from $\mathcal{V}$ and let x_1 and y_1 be points of X_1. Let $A_1 \in \mathcal{Y}_1$ and $A_2 \in \mathcal{Y}_2$ be the lines through x_1 and $B_1 \in \mathcal{Y}_1$ and $B_2 \in \mathcal{Y}_2$ be the lines through y_1. We denote by C_1 and D_1 the line from $\mathcal{Y}_2$ which is incident with $A_1 \cap X_2$ respectively with $B_1 \cap X_2$; by C_2 and D_2 we denote the line from $\mathcal{Y}_1$ which is incident with $A_2 \cap X_2$ respectively with $B_2 \cap X_2$. We call a configuration consisting of the lines $X_1, X_2, A_1, A_2, B_1, B_2, C_1, C_2, D_1, D_2$ and of the points $x_1, y_1, A_1 \cap X_2, B_1 \cap X_2, A_2 \cap X_2, B_2 \cap X_2, C_1 \cap C_2$ and $D_1 \cap D_2$ a non-closed Bol configuration and say that the 3-net $\mathcal{N}$ satisfies the Bol condition with respect to the pencil of vertical lines (or that the non-closed Bol configuration can be closed with respect to the pencil of vertical lines) if $C_1 \cap C_2$ and $D_1 \cap D_2$ belong to the same vertical line X_3. If a 3-net satisfies this condition we call it a **Bol net**. It is well known that the coordinate loops of a 3-net $\mathcal{N}$ are left Bol loops if and only if $\mathcal{N}$ is a Bol net. The coordinate loops of a 3-net $\mathcal{N}$ are Moufang loops if and only if $\mathcal{N}$ satisfies the Bol conditions for two (and then for any) pencils of lines. In this case $\mathcal{N}$ is called a **Moufang net**.

First we consider the topological projectivity group $\widehat{\Pi}$ as well as the groups topologically generated by the left, right and all translations respectively in the case of a Moufang loop in the category $\mathcal{C}$.

THEOREM 1.1: *Let L be a differentiable connected Moufang loop (of class C^∞). Then:*

(i) *$\widehat{\Pi}$ is a Lie group.*

(ii) *The closure $\widehat{\mathfrak{M}}$ of the group $\mathfrak{M}$ generated by the left and right translations of L in $\widehat{\Pi}$ is a connected Lie group.*

(iii) *The closures $\widehat{\mathfrak{L}}$ and $\widehat{\mathfrak{R}}$ of the groups $\mathfrak{L}$ and $\mathfrak{R}$ generated by the left translations and right translations, respectively, are closed connected Lie subgroups of $\widehat{\mathfrak{M}}$.*

Proof: According to Theorem 6.1 in [2], §6, the group $\widehat{\Pi}$ is topologically generated by the sets $\Lambda = \{\lambda_x, x \in L\}$, $P = \{\rho_x, x \in L\}$ and by the mapping $\iota\colon x \to x^{-1}$.

The connected component $\widehat{\Pi}^{(1)}$ of the identity in $\widehat{\Pi}$ is generated by Λ and P and has index at most 2 in $\widehat{\Pi}$ since $\iota\Lambda\iota = P$. The stabilizer $\widehat{\Pi}^{(1)}_e$ of the unit point $e \in L$ in the group $\widehat{\Pi}^{(1)}$ is the topological closure of the inner mapping group in $\widehat{\Pi}^{(1)}$ (cf. [4], Lemma 1.2, p. 61). From Lemma 3.2 in [4] p. 117, it follows that $\widehat{\Pi}^{(1)}_e$ is a closed subgroup of the group of pseudoautomorphisms of L. But this group is a Lie group by Theorem 10.21 in [2], p. 55. Since $\widehat{\Pi}^{(1)}/\widehat{\Pi}^{(1)}_e$ is diffeomorphic to L the group $\widehat{\Pi}^{(1)}$ is a manifold and hence $\widehat{\Pi}$ is a Lie group (cf. [22]).

The assertions (ii) and (iii) follow from (i) immediately. ∎

PROPOSITION 1.2: *If L is a topological loop such that the group $\widehat{\mathfrak{M}}$ topologically generated by all left and right translations of L is a connected non-abelian Lie-simple compact Lie group, then L is the classical Moufang loop of the Cayley numbers of norm 1 on S^7 or its factor loop on the 7-dimensional real projective space P^7.*

Proof: From a result of Scheerer [29] it follows that the universal covering loop $\widehat{L}$ of L is realized on a sphere S^n with $n = 1, 3, 7$; moreover for $n = 1, 3$ the loop L is a group and $\widehat{\mathfrak{M}}$ cannot be a non-abelian Lie-simple group. Let now be $n = 7$. Since L has an invariant uniformity (cf. [12]) L must be one of the two loops mentioned in the assertion (cf. [11]). ∎

We remark that for the classical Moufang loops on S^7 and P^7 the groups generated by all multiplications are the orthogonal groups $\mathrm{SO}_8(\mathbb{R})$ and $\mathrm{PSO}_8(\mathbb{R})$, respectively. The corresponding groups of projectivities are the groups $\mathrm{O}_8(\mathbb{R})$ and $\mathrm{PO}_8(\mathbb{R})$ respectively, which are not connected.

THEOREM 1.3: *Let L be a compact connected Lie Moufang loop. Then:*

(i) *The groups topologically generated by the left translations, by the right translations and by all translations respectively are compact connected Lie transformation groups.*
(ii) *The topological projectivity group $\widehat{\Pi}$ of L is a compact Lie transformation group.*

Proof: Let $\mathfrak{L}$ be the Malcev algebra of L. Then according to the results of E.N. Kuz'min (cf. [14], [16] and [10], pp. 250–251) $\mathfrak{L}$ has a unique largest solvable ideal, the radical $\mathfrak{R}$ of $\mathfrak{L}$, such that $\mathfrak{L}$ is a semidirect product $\mathfrak{R}\mathfrak{s}$ where $\mathfrak{s}$ is a semisimple Malcev algebra. Let R be the solvable subloop of L which corresponds to $\mathfrak{R}$ and $X, Y \in \mathfrak{R}$. In the Malcev algebra $\mathfrak{R} \subset \mathfrak{L}$ the elements X, Y generate a Lie subalgebra. The closure of the corresponding closed Lie subgroup in R is a compact solvable Lie subgroup and hence a torus group. From this follows that $\mathfrak{R}$ is a commutative Malcev algebra and the subloop R of L is a torus group. Now the loop L is a product of the torus group R with a semisimple Lie Moufang loop S such that R is a normal subgroup of L and $R \cap S$ is finite. Hence, the semisimple Moufang loop S is an almost direct product of loops of the following type: compact connected Lie simple groups and classical Lie Moufang loops on S^7 and P^7 (cf. [14], [17] and [10], pp. 250–251). ∎

Now we show that the theory of symmetric spaces can be applied in differentiable 3-nets to give an alternative proof of Theorem 1.1 and for a differentiable

Bol loop to prove the Lie nature of the group (topologically) generated by the left translations.

PROPOSITION 1.4: *Let $\mathcal{N}$ be a 3-net, L a line in $\mathcal{N}$ and x a point on L. We denote by K_x^p ($p \in \mathbb{Z}_2$) the other two lines through x. The projectivities σ_x^L having as irreducible representations*

$$[L, \mathfrak{X}_1, K_x^p]\,[K_x^p, \mathfrak{X}_2, K_x^{p+1}]\,[K_x^{p+1}, \mathfrak{X}_3, L]$$

satisfy the following properties:

(i) $\sigma_x^L(x) = x$;

(ii) $(\sigma_x^L)^2 = \mathrm{id}$ *if and only if in $\mathcal{N}$ the hexagonal condition for the point x holds;*

(iii) *if $\alpha = [L, \mathfrak{D}, L']$ with $L, L' \notin \mathfrak{D}$ then $\alpha^{-1}\sigma_x^L\alpha = \sigma_{x\alpha}^{L'}$ for all σ_x^L if and only if $\mathcal{N}$ satisfies the Bol condition for the pencil $\mathfrak{D}$ of parallel lines;*

(iv) *if $\mathcal{N}$ satisfies the Bol condition for the pencil $\mathfrak{D}$ then $\sigma_x^L\sigma_y^L\sigma_x^L = \sigma_{y\sigma_x^L}^L$ for all lines $L \notin \mathfrak{D}$ and all points $x, y \in L$.*

Proof: The properties (i) and (ii) are trivial. The identity $\alpha^{-1}\sigma_x^L\alpha = \sigma_{x\alpha}^{L'}$ holds if and only if for every $z \in L$ the following two quadruples of points:

$$\{x, z, z[L, \mathfrak{X}_1, K_x^p], z[L, \mathfrak{X}_1, K_x^p][K_x^p, \mathfrak{X}_2, K_x^{p+1}]\}$$

and

$$\{x\alpha, z\alpha, z\alpha[L', \mathfrak{X}_1', K_{x\alpha}^q], z\alpha[L', \mathfrak{X}_1', K_{x\alpha}^q]\,[K_{x\alpha}^q, \mathfrak{X}_2', K_{x\alpha}^{q+1}]\}\ (p, q \in \mathbb{Z}_2)$$

together with the joining lines form the Bol configuration with respect to $\mathfrak{D}$ and (iii) is proved.

Now we show (iv). Since for every line $G \notin \mathfrak{D}$ and every point $t \in G$ one has $(\sigma_t^G)^2 = \mathrm{id}$, any projectivity σ_t^G can be represented irreducibly as

$$[G, \mathfrak{Y}, D_t]\,[D_t, \mathfrak{X}, K_t]\,[K_t, \mathfrak{D}, G]\ ,$$

where D_t is the line through t contained in the pencil $\mathfrak{D}$. Let L' be the line parallel to L through the point $y[L, \mathfrak{Y}, D_x]$ and $\beta = [L, \mathfrak{D}, L']$. Now using (ii) and (iii) we have

$$\sigma_x^L\sigma_y^L\sigma_x^L\sigma_{y\sigma_x^L}^L = (\sigma_x^L)^{-1}\sigma_y^L\beta(\sigma_{x\beta}^{L'})^{-1}\beta^{-1}\beta\sigma_{y\sigma_x^L\beta}^{L'}\beta^{-1}\ .$$

The last mapping has a representation

$$\begin{aligned}&[L,\mathfrak{D},K_x][K_x,\mathfrak{X},D_x][D_x,\mathfrak{Y},L][L,\mathfrak{Y},D_y][D_y,\mathfrak{X},K_y]\\ &\cdot[K_y,\mathfrak{D},L][L,\mathfrak{D},L'][L',\mathfrak{D},K_{x\beta}][K_{x\beta},\mathfrak{X},D_{x\beta}][D_{x\beta},\mathfrak{Y},L']\\ &\cdot[L',\mathfrak{Y},D_{y\sigma_x^L\beta}][D_{y\sigma_x^L\beta},\mathfrak{X},K_{y\sigma_x^L\beta}][K_{y\sigma_x^L\beta},\mathfrak{D},L'][L',\mathfrak{D},L]\\ =&[L,\mathfrak{D},K_x][K_x,\mathfrak{X},D_x][D_x,\mathfrak{Y},D_y][D_y,\mathfrak{X},K_y][K_y,\mathfrak{D},K_{x\beta}][K_{x\beta},\mathfrak{X},D_{x\beta}]\\ &\cdot[D_{x\beta},\mathfrak{Y},D_{y\sigma_x^L\beta}][D_{y\sigma_x^L\beta},\mathfrak{X},K_{y\sigma_x^L\beta}][K_{y\sigma_x^L\beta},\mathfrak{D},L]\\ =&[L,\mathfrak{D},K_x][K_x,\mathfrak{X},D_x][D_x,\mathfrak{Y},D_y][D_y,\mathfrak{X},D_x][D_x,\mathfrak{Y},D_{y\sigma_x^L}]\\ &\cdot[D_{y\sigma_x^L},\mathfrak{X},K_x][K_x,\mathfrak{D},L],\end{aligned}$$

since $K_{x\beta}=K_y$, $D_{x\beta}=D_x$, $D_{y\sigma_x^L\beta}=D_{y\sigma_x^L}$, $K_{y\sigma_x^L\beta}=K_x$ and hence

$$[K_y,\mathfrak{D},K_{x\beta}]=[K_y,\mathfrak{D},K_y].$$

We denote:

$$\begin{aligned}\alpha_1=[L,\mathfrak{D},K_x][K_x,\mathfrak{X},D_x],\quad &\alpha_2=\alpha_1[D_x,\mathfrak{Y},D_y],\\ \alpha_3=\alpha_2[D_y,\mathfrak{X},D_x]\quad\text{and}\quad &\alpha_4=\alpha_3[D_x,\mathfrak{Y},D_{y\sigma_x^L}].\end{aligned}$$

The points $y\sigma_x^L\alpha_1$, $y\sigma_x^L\alpha_2=y$, $y\sigma_x^L\alpha_3=x$, $y\sigma_x^L\alpha_4$, $z\alpha_1$, $z\alpha_2$, $z\alpha_3$, $z\alpha_4$ $(z\in L)$ and their joining lines form a non-closed Bol configuration. Since the 3-net $\mathcal{N}$ satisfies the Bol condition with respect to the pencil $\mathfrak{D}$, the point $z\alpha_4$ is contained in the line through $z\alpha_1$ which is parallel to L. Hence $z\alpha_4[D_{y\sigma_x^L},\mathfrak{X},K_x]\,[K_x,\mathfrak{D},L]=z$, which means $\sigma_x^L\sigma_y^L\sigma_x^L\sigma_{y\sigma_x^L}^L=\mathrm{id}$ and (iv) is true. ∎

Let L be a differentiable loop of the class C^∞. Then the associated 3-net $\mathcal{N}=L\times L$ is a differentiable 3-net and the perspectivities $[G,\mathfrak{X},G']$ are differentiable mappings between the lines G and G'.

According to [18], p. 63, a symmetric space is a manifold M with a differentiable multiplication $\mu: M\times M\to M$ having the following properties:

(i) $\mu(x,x)=x$ for all $x\in M$,
(ii) $\mu(x,\mu(x,y))=y$ for all $x,y\in M$,
(iii) $\mu(x,\mu(y,z))=\mu(\mu(x,y),\mu(x,z))$ for all $x,y,z\in M$,
(iv) every $x\in M$ has a neighbourhood U such that $\mu(x,y)=y$ implies $y=x$ for all $y\in U$.

PROPOSITION 1.5: *In a differentiable 3-net $\mathcal{N}$ satisfying the Bol condition with respect to the pencil $\mathfrak{D}$ of lines every line $G\notin\mathfrak{D}$ is a symmetric space with the multiplication $\mu(x,y)=y\sigma_x^G$ and every perspectivity $[G,\mathfrak{D},G']$ is an isomorphism from the symmetric space G onto the symmetric space G'.*

If $e \in G$ is the unit of a coordinate loop on the horizontal line G, then $y\sigma_e^G = y^{-1}$.

If $\mathcal{N}$ is a Moufang net, then every line G of $\mathcal{N}$ is a symmetric space with respect to the multiplication μ and every projectivity $\pi : G \to G'$ is an isomorphism of symmetric spaces.

Proof: The properties (i)–(iii) of symmetric spaces follow from Proposition 1.4. Let e be an arbitrary point of a line $G \notin \mathfrak{D}$ and L_G^e be the differentiable coordinate Bol loop on G with the unit element e. Then we have $y\sigma_e^G = y^{-1}$ and $y\sigma_x^G = y$ if and only if $y = y^{-1}$. In the Bol loop L_G^e there exists a suitable neighbourhood U such that every element of U is contained in precisely one local 1-parameter subgroup of L_G^e (cf. [25]). Hence condition (iv) in the definition of symmetric spaces is satisfied. From (iii) of Proposition 1.4 follows that any perspectivity $[G, \mathfrak{D}, G']$ and in the case of Moufang loops every projectivity is an isomorphism of symmetric spaces. ∎

The left, right and middle nucleus, respectively, of a loop L are the subgroups of L which are defined in the following way:

$$N_l = \{u \,|\, ux \cdot y = u \cdot xy\},\ N_r = \{w \,|\, x \cdot yw = xy \cdot w\} \text{ and } N_m = \{v \,|\, xv \cdot y = x \cdot vy\}$$

for every $x, y \in L$.

The following statements show how the right nucleus of a loop can be recognized within the collineation group of the associated 3-net.

LEMMA 1.6: *Let L be a loop and $\mathcal{N}(L)$ be the associated 3-net. Then the group Θ consisting of the collineations of $\mathcal{N}(L)$ which preserve all the 3 pencils of lines and which leave every vertical line invariant is the set of mappings of the form $(x, y) \mapsto (x, y\rho_a) : L \times L \to L \times L$, where a is in the right nucleus N_r of L and $\rho_a : y \mapsto ya$.*

Proof: If $\theta \in \Theta$ then Θ leaves every vertical line invariant and maps horizontal lines onto horizontal lines; hence it can be written as $\theta : (x, y) \mapsto (x, y^\alpha)$.

The transversal line through a point (x, y) contains the point $(xy, 1)$, too. It follows that $xy \cdot 1^\alpha = x \cdot y^\alpha$. We denote $a = 1^\alpha$. If we put $x = 1$ we obtain $ya = y^\alpha$ or $\alpha = \rho_a$. It follows that $xy \cdot a = x \cdot ya$ and $a \in N_r$. Clearly, this condition is sufficient too. ∎

Let $\mathcal{N}$ be a Bol net. Then to every vertical line G there exists an involutory collineation τ_G of $\mathcal{N}$ fixing G pointwise. If p is a point of $\mathcal{N}$ then $p\tau_G$ is the intersection of the two non-vertical lines meeting the line G in the same points as the two non-vertical lines through p. The involutory collineation τ_G is called the **Bol reflection** on G.

Since the collineations τ_G $(G \in \mathcal{V})$ preserve the pencil $\mathcal{V}$ of vertical lines we can consider the mapping $\widetilde{\tau}_G$ which is induced by the collineation τ_G on the pencil $\mathcal{V}$.

LEMMA 1.7: *For the map $\alpha : \tau_G \to \widetilde{\tau}_G$ there holds*

$$\alpha(\tau_{G_1} \circ \tau_{G_2}) = \alpha(\tau_{G_1}) \cdot \alpha(\tau_{G_2}).$$

Hence the group $\widetilde{\Gamma}$ generated by the mappings $\widetilde{\tau}_G$ is a homomorphic image of the collineation group Γ generated by the collineations τ_G. The kernel of this homomorphism α is isomorphic to a subgroup of the right nucleus N_r of a coordinate loop L of the Bol net $\mathcal{N}$. The image of any stabilizer of a point p of $\mathcal{N}$ in Γ under $\alpha : \Gamma \to \widetilde{\Gamma}$ is contained in the stabilizer of the vertical line through p in $\widetilde{\Gamma}$. If Γ_0 denotes the subgroup of Γ consisting of products of an even number of reflections τ_G, then the stabilizer of a point $p \in \mathcal{N}$ in Γ_0 is contained in the intersection of the stabilizers in Γ_0 of the horizontal and of the vertical line through p.

Proof: The multiplicative property of the mapping $\alpha : \Gamma \to \widetilde{\Gamma}$ is a consequence of the fact that the collineations τ_G $(G \in \mathcal{V})$ preserve the pencil of vertical lines. From the previous lemma it follows that the kernel of α is isomorphic to a subgroup of N_r. Since Γ preserves the pencil of vertical lines and Γ_0 leaves each of the 3 pencils of lines invariant, Lemma 1.7 is proved. ∎

If H is a horizontal line through the origin of the coordinate system and $[\mathcal{V}, H] : H \to \mathcal{V}$ is the perspectivity from H to $\mathcal{V}$ then we have $[\mathcal{V}, H]^{-1} \circ \widetilde{\tau}_G \circ [\mathcal{V}, H] = \sigma^H_{G\cap H}$. Hence we obtain the

COROLLARY 1.8: *In a Bol 3-net $\mathcal{N}$ the transformation group $\widetilde{\Gamma}$ is isomorphic to the group Σ generated by the reflections σ^H_x, $x \in H$. The stabilizer of a vertical line $G \in \mathcal{V}$ in $\widetilde{\Gamma}$ is isomorphic to the stabilizer of $G \cap H$ in Σ.*

In the next theorem we investigate the group generated by all Bol reflections in a differentiable Bol 3-net and its relations to the group generated by the left translations.

THEOREM 1.9: *Let L be a connected differentiable Bol loop and $\mathcal{N}(L)$ the corresponding 3-net. Let Γ be the collineation group of $\mathcal{N}(L)$ which is (topologically) generated by the reflections τ_G $(G \in \mathcal{V})$.*

(i) *The Lie transformation group Σ generated by the reflections σ_x, $x \in L$, in the symmetric space S on the horizontal line L is a homomorphic image* *of Γ. The kernel D of the homomorphism β which assigns to any element of Γ its action on S is a closed subgroup of the right nucleus N_r of L and the image of the stabilizer of the origin $(1,1)$ of $\mathcal{N}(L)$ in Γ under β is contained in the stabilizer of $1 \in L$ in Σ. The Lie transformation group $\widetilde{\Gamma}$ induced by Γ on the pencil $\mathcal{V}$ of vertical lines is isomorphic to Σ and operates transitively on $\mathcal{V}$.*

(ii) *The group Γ is a Lie transformation group on $\mathcal{N}(L)$.*

(iii) *The group $\widehat{\mathcal{L}}$ (topologically) generated by the left translations of L is a Lie transformation group which is a homomorphic image of the connected component Γ_0 of the collineation group Γ. There is a canonical epimorphism $\Phi : \Gamma_0 \to \widehat{\mathcal{L}}$ the kernel K of which is isomorphic to a closed subgroup of the left nucleus N_l of L. The image of the stabilizer of the origin of $\mathcal{N}(L)$ in Γ_0 under Φ is contained in the stabilizer of $1 \in L$ in $\widehat{\mathcal{L}}$.*

Proof: The group Σ on the symmetric space S is a Lie transformation group according to [18], Theorem 2.7, p. 88, and Theorem 1.6, p. 129. The assertion (i) follows from Lemma 1.7, Corollary 1.8, Lemma 1.6 and from the fact that the kernel D is a closed subgroup in the topological group Γ; the group D consists of the elements of Γ fixing every vertical line. According to Lemma 1.6 the group D is isomorphic to a closed subgroup of N_r.

Since every closed subgroup of a differentiable loop is a Lie group the assertion (ii) follows from the fact that any extension of a Lie group by a Lie group is a Lie group.

According to [5], Theorem 3.1, there exists a canonical epimorphism $\Phi : \Gamma_0 \to \widehat{\mathcal{L}}$; the kernel K of Φ is a closed subgroup of Γ consisting of such elements of Γ_0 which leave every horizontal line invariant (cf. [5], p. 66 and p. 69); moreover K is isomorphic to a closed subgroup of the left nucleus N_l of L. The group $\widehat{\mathcal{L}}$ is the group which is induced by Γ_0 on the pencil of horizontal lines. ∎

Alternative proof of Theorem 1.1: Since the projectivities of a line G are automorphisms of the symmetric space on G (cf. Proposition 1.5) the statements follow from the fact that $\widehat{\Pi}$ is a closed subgroup of the automorphism group Γ of the symmetric space on a line G, and Γ is a Lie group (cf. [18], Theorem 2.7, p. 88 and Theorem 1.6, p. 129). ∎

Now we prove the

THEOREM 1.10: *Every differentiable connected Moufang loop L is analytic.*

Proof: Let $\mathcal{N} = L \times L$ be the 3-net associated with L. The structure of the symmetric space $\Sigma(G)$ given by the reflections σ_x^G, $x \in G$ on the line G provides

G with an analytic structure $\rho(G)$. Identifying G with L the analytic structure $\rho(L)$ on L defines the analytic product structure on the net manifold $\mathcal{N} = H_1 \times V_1$, where H_1 is the horizontal line and V_1 is the vertical line through the origin $(1, 1)$. The left translation λ_x on $H_1 = L$ is given by

$$\lambda_x = [H_1, \mathfrak{T}, V_1]\,[V_1, \mathfrak{H}, V_x]\,[V_x, \mathfrak{T}, H_1], \quad x \in L,$$

where V_x is the vertical line through $(x, 1)$, $\mathfrak{T}$ is the family of transversal and $\mathfrak{H}$ the family of horizontal lines. Since in the 3-net $\mathcal{N}$ all of the three Bol conditions are satisfied, every projectivity between lines G and G' of $\mathcal{N}$ preserves the analytic structure given by the analytic spaces $\Sigma(G)$ and $\Sigma(G')$ (cf. Proposition 1.5). If we provide the family $\mathfrak{V}$ of vertical lines with the analytic structure given by the intersections of vertical lines V_x with H_1 then the map $\tau : x \mapsto \lambda_x$ is an analytical map from $L = H_1$ into the Lie group $\widehat{\mathcal{L}}$ topologically generated by the left translations of L which acts on L as an analytic transformation group such that the mapping $\omega : \widehat{\mathcal{L}} \times L \to L : (\gamma, x) \mapsto x\gamma$ is analytic. From this follows that the composed map $(x, y) \mapsto (x, y)(\tau, \mathrm{id})\omega = x \cdot y$ is analytic. Since the map $\sigma_1^L : x \mapsto x^{-1}$ is the reflection on the point 1 in the symmetric space induced on the horizontal line through the origin, it is analytic. Since the Moufang loop has the inverse property, $x \backslash y = x^{-1}y$ and $x/y = xy^{-1}$ are analytic. Hence the assertion is proved. ∎

For differentiable (left) Bol loops we are not able to prove the previous theorem but it is possible to obtain partial results in this direction. In the next theorem we show that in a Bol loop on a differentiable manifold the smoothness of some operations forces the differentiability of all operations of the loop. If r is a natural number or $r = \infty$ then C^r denotes the differentiability class of order r; if $r = \omega$ then C^r denotes real analyticity.

THEOREM 1.11:

(i) *Let L be a Bol loop on a C^r-manifold such that the multiplication $(x, y) \mapsto x \cdot y$ is of the class C^r. Then the operations $(x, y) \mapsto x \backslash y$ and $(x, y) \mapsto y/x$ are both of the class C^r and L is a Bol loop in the C^r-category ($r \in \mathbb{N} \cup \{\infty\} \cup \{\omega\}$).*

(ii) *Let G be a connected Lie group and H be a subgroup of G such that $\sigma : G/H \to G$ is a section of the class C^r for the canonical projection $\pi : G \to G/H$. Let $x \cdot y = \sigma\big(\pi(xy)\big)$ for $x, y \in \sigma(G/H)$ be a loop L satisfying $xyx \in \sigma(G/H)$. Then L is a Bol loop of the class C^r.*

Proof: (i) According to Proposition 1.5 the inverse mapping $x \mapsto x^{-1}$ in L coincides with the involutory mapping $x \mapsto x\sigma_e^L$ in the symmetric space S on the horizontal line L in the 3-net $L \times L$ with the origin (e,e). This is a real analytic mapping in the symmetric space S and consequently it is of the class C^r on the loop manifold L.

Since L satisfies the left inverse property we have $x\backslash y = x^{-1} \cdot y$ and hence $(x,y) \mapsto x\backslash y$ is of the class C^r. If $y = b/a$ then using the Bol identity we can write (cf. [6], Lemma 2)

$$y = a^{-1}\cdot(a\cdot y) = a^{-1}\Big(a\cdot\big(y\cdot(a\cdot a^{-1})\big)\Big) = a^{-1}\cdot\Big(\big(a\cdot(y\cdot a)\big)\cdot a^{-1}\Big) = a^{-1}\cdot\big((a\cdot b)\cdot a^{-1}\big)$$

which says that the mapping $(b,a) \mapsto b/a$ is a composition of mappings of the class C^r.

(ii) The group topologically generated by the left translations of L is the subgroup of G which is generated by $\sigma(G/H)$ and $\sigma(G/H)$ consists of the left translations of L. The Bol identity is equivalent to the fact that for any $x, y \in \sigma(G/H)$ we have $xyx \in \sigma(G/H)$. Since the mappings π and σ are of the class C^r the multiplication $(x,y) \mapsto x \cdot y$ is of the class C^r. Hence (ii) follows from (i). ∎

At the end of this section we consider classes of Moufang loops in which isotopic loops are isomorphic.

PROPOSITION 1.12: *Every loop isotopic to a loop L is isomorphic to L (i.e. L is a G-loop) if and only if the direction preserving collineation group Ω operates point transitively on the 3-net $\mathcal{N}$ associated with L.*

Proof: Since the multiplications on coordinate loops of $\mathcal{N}$ can be defined geometrically, the direction preserving collineation group Ω, which operates point transitively, induces isomorphisms between principal isotopic loops. Conversely, every isomorphism between principal isotopic loops can be extended to a direction preserving collineation.

THEOREM 1.13: *A Moufang loop L is a G-loop if and only if each element of L is a companion of a pseudoautomorphism of L.*

The assertion is a consequence of Proposition 1.12 and of Theorem 10.17 in [2], p. 50.

THEOREM 1.14: *A Moufang loop L is a G-loop if one of the following conditions holds:*

(i) *every element of L is a commutator;*

(ii) *the map $x \mapsto x^3$ is surjective;*

(iii) *the normal subloop NL^3 of L which is the product of the nucleus N of L with the normal subloop L^3 generated by the cubes of the elements of L is equal to L.*

The statement is a consequence of Proposition 1.12 and of Corollary 10.18 in [2], p. 51.

THEOREM 1.15: *Every connected differentiable Moufang loop L is a G-loop.*

The assertion is a consequence of Proposition 1.12 and Theorem 10.25 (1) in [2], p. 56.

§2. Bol loops, their cores and left distributive quasigroup

In this section we discuss mainly the connection between isotopy classes of Bol loops and certain quasigroups, following the ideas of Belousov [3]. In the 3-net $\mathcal{N}$ associated with a Bol loop the Bol condition is satisfied with respect to the family of vertical lines.

Definition 2.1: cf. [3], p. 210. Let L be a Bol loop. The (left) core of the loop L is the quasigroupoid consisting of the elements of L under the operation "+" defined by

$$x + y = x(y^{-1}x).$$

PROPOSITION 2.2: *If L is a Bol loop then $(x+y)^{-1} = x^{-1} + y^{-1}$ for all $x, y \in L$.*

Proof: The assertion is true if $(x \cdot y^{-1}x)(x^{-1} \cdot yx^{-1}) = 1$ holds. Since L is a Bol loop one has $(x \cdot y^{-1}x)z = x(y^{-1} \cdot xz)$ and taking $z = x^{-1} \cdot yx^{-1}$ we obtain the statement. ∎

PROPOSITION 2.3: *A Bol loop L is a Moufang loop if and only if one of the following two conditions is satisfied:*

(i) $(x + y)z = xz + yz$,

(ii) $z(x + y) = zx + zy$,

for all $x, y, z \in L$.

Proof: (i) Let $\mathcal{N}$ be the net associated with the loop L. Let G be the horizontal line through the origin and let G' be the horizontal line through the point $(1, z)$. By $\mathfrak{V}$, $\mathfrak{T}$ and $\mathfrak{H}$ we denote the families of vertical, transversal, and horizontal lines, respectively. We have $(y,1)\sigma^G_{(x,1)} = (x + y, 1)$. From Proposition 1.4 (iii) we know that

$$\begin{aligned}(y,z)\sigma^{G'}_{(x,z)} &= (y,z)[G',\mathfrak{V},G]\sigma^{G}_{(x,z)[G',\mathfrak{V},G]}[G,\mathfrak{V},G'] \\ &= (y,1)\sigma_{(x,1)}[G,\mathfrak{V},G'] = (x+y,z)\end{aligned}$$

since $\mathcal{N}$ satisfies the Bol condition with respect to the family of vertical lines. On the one hand we have $(xz + yz, 1) = (yz, 1)\sigma^G_{(xz,1)}$; on the other hand we can write

$$((x+y)z, 1) = (x+y, z)[G', \mathfrak{T}, G] = (y, z)\sigma^{G'}_{(x,z)}[G', \mathfrak{T}, G]$$
$$= (yz, 1)[G, \mathfrak{T}, G']\sigma^{G'}_{(x,z)}[G', \mathfrak{T}, G].$$

It follows that $xz + yz = (x+y)z$ if and only if the Bol condition with respect to the family $\mathfrak{T}$ is satisfied.

The proof of the statement (ii) is analogous. ∎

Definition 2.4: A permutation α of a Bol loop L onto itself is called a **semiautomorphism** of L if

$$(x \cdot yx)^\alpha = x^\alpha \cdot y^\alpha x^\alpha \quad \text{and} \quad 1^\alpha = 1$$

for all $x, y \in L$.

THEOREM 2.5: *Let L be a Bol loop and $(L, +)$ the core of L. Then*

(i) *the semiautomorphisms of L are the automorphisms of $(L, +)$ leaving $1 \in L$ fixed.*
(ii) *$(L, +)$ is isomorphic to the quasigroupoid on the section $\{\lambda_x,\ x \in L\}$ with respect to the multiplication $\lambda_x + \lambda_y := \lambda_x\lambda_y^{-1}\lambda_x$.*
(iii) *Principal isotopic Bol loops have the same core.*
(iv) *In $(L, +)$ the left distributivity law $x+(y+z) = (x+y)+(x+z)$ $(x, y, z \in L)$ holds.*
(v) *In $(L, +)$ one has $x + (x + y) = y$ for all $x, y \in L$.*

Proof: The statement (i) follows in the same way as the assertion (ii) in the Theorem 5.1 in [4], pp. 121–122. Since in a left Bol loop we have $\lambda_{x\cdot yx} = \lambda_x\lambda_y\lambda_x$ and $\lambda_{y^{-1}} = \lambda_y^{-1}$, one obtains $\lambda_{x+y} = \lambda_x + \lambda_y$ which is the property (ii).

The statement (iii) follows from the fact that the core is invariant with respect to perspectivities with vertical direction; on the same horizontal line of a Bol net the definition of the core is independent of the choice of the unit element. (For another proof cf. [3], Theorem 11.8.)

The proof of statement (iv) is given in [3], Theorem 1.9, pp. 211–213. Using Proposition 1.4 we obtain another proof:

$$x + (y + x) = z\sigma^L_y\sigma^L_x = z\sigma^L_x\sigma^L_{y\sigma^L_x} = (x+y) + (x+z).$$

Proposition 1.4 (ii) gives the property (v). ∎

PROPOSITION 2.6: *Let L be a Bol loop. Its core $(L,+)$ is a quasigroup if and only if the mapping $x \mapsto x^2$ is a permutation of L. If this condition is satisfied then $(L,+)$ is isotopic to a Bol loop.*

Proof: See [3], Theorems 11.10 and 11.11, pp. 213–215. ∎

We denote by $L^+(a)$ and $R^+(a)$ respectively the mappings $x \mapsto a+x$ and $x \mapsto x+a$ in the core $(L,+)$ of a Bol loop L.

THEOREM 2.7: *Let L be a Bol loop such that the mapping $x \mapsto x^2$ is a permutation of L. Then the core $(L,+)$ is isotopic to the loops $L(\frac{1}{2},a)$ which are defined on the elements of L by the multiplication*

$$x *_a y = xR^+(a)^{-1} + yL^+(a)^{-1} \quad \text{for all } x,y \in L \text{ and a fixed } a \in L.$$

The unit element of the loop $L(\frac{1}{2},a)$ is a; for $a=1$, the unit element of L, we have $x *_1 y = x^{\frac{1}{2}} \cdot yx^{\frac{1}{2}}$.

Any loop $L(\frac{1}{2},a)$ has the following properties:

(i) *It is a Bol loop.*

(ii) *It has the automorphic inverse property, i.e. $(x *_a y)^{-1} = x^{-1} *_a y^{-1}$ for all $x,y \in L$.*

(iii) *It is a left A-loop, i.e. the left inner mappings $z \mapsto z\lambda_y\lambda_x\lambda_{xy}^{-1}$ are automorphisms of L.*

If L and L' are isotopic Bol loops then for any $a \in L$ and $a' \in L'$ the loops $L(\frac{1}{2},a)$ and $L'(\frac{1}{2},a')$ are isomorphic.

Proof: Since $a+a=a$ the unit element of $L(\frac{1}{2},a)$ is a. By definition any loop $L(\frac{1}{2},a)$ is isotopic to the core $(L,+)$. Since any loop isotopic to a Bol loop is Bol we obtain from Proposition 2.6 that any loop $L(\frac{1}{2},a)$ is a Bol loop.

Now we prove that the loops $L(\frac{1}{2},a)$ and $L(\frac{1}{2},b)$ $a,b \in L$ are isomorphic. An isomorphism from $L(\frac{1}{2},a)$ to $L(\frac{1}{2},b)$ is given by the left translation $\lambda^*_{(a)}(b)\colon y \mapsto b *_a y$. We have $\lambda^*_{(a)}(b) = L^+(a)L^+(bR^+(a)^{-1})$ since $L^+(a)^{-1} = L^+(a)$. We put $c = bR^+(a)^{-1}$ or equivalently $c+a=b$. Then we can write $\lambda^*_{(a)}(b) = L^+(a)L^+(c)$. On one hand we have $(x *_a y)L^+(a)L^+(c) = [xR^+(a)^{-1} + yL^+(a)]L^+(a)L^+(c)$. The left distributivity law for the core $(L,+)$ implies that the last expression is equal to $xR^+(a)^{-1}L^+(a)L^+(c) + yL^+(c)$. On the other hand we compute $x\lambda^*_{(a)}(b) *_b y\lambda^*_{(a)}(b) = xL^+(a)L^+(c)R^+(b)^{-1} + yL^+(a)L^+(c)L^+(b)$. Since for any $q \in L$ one has $qL^+(c)L^+(b) = b+(c+q) = (c+a)+(c+q) = c+(a+q) = qL^+(a)L^+(c)$ we obtain $yL^+(a)L^+(c)L^+(b) = yL^+(a)L^+(a)L^+(c) = yL^+(c)$. Hence the mapping $\lambda^*_{(a)}(b)$ is an isomorphism if and only if

$$R^+(a)^{-1}L^+(a)L^+(c) = L^+(a)L^+(c)R^+(b)^{-1}$$

or

$$L^+(a)L^+(c)R^+(b) = R^+(a)L^+(a)L^+(c).$$

For any $u \in L$ this means

$$[c + (a + u)] + b = c + [a + (u + a)].$$

We prove that this identity is satisfied in $(L, +)$. In fact if we remember $c + a = b$ then on the left side of this identity we have

$$[c + (a + u)] + b = [b + (c + u)] + b = [b + (c + u)] + (b + b)$$
$$= b + [(c + u) + b] = (c + a) + [c + (u + a)] = c + [a + (u + a)]$$

which is the right side of our identity. Hence all loops $L(\frac{1}{2}, a)$ are isomorphic with respect to the family of isomorphisms $\lambda^*_{(a)}(b)$. It follows that the loops $L(\frac{1}{2}, t)$ ($t \in L$) corresponding to different coordinate loops of the 3-net $\mathcal{N}$ associated with L are isomorphic since the core $(L, +)$ of L is the same for all coordinate loops of the 3-net $\mathcal{N}$ (cf. Theorem 2.5 (iii)). Hence the last statement of the theorem is proved.

Since all loops $L(\frac{1}{2}, a)$ are isomorphic we prove (ii) for the loop $L(\frac{1}{2}, 1)$. In this case $x *_1 y = x^{\frac{1}{2}} \cdot yx^{\frac{1}{2}}$. Since $x *_1 x^{-1} = x^{\frac{1}{2}} \cdot x^{-1}x^{\frac{1}{2}} = 1$, the inverse of x is x^{-1} and $x *_1 y * (x^{-1} *_1 y^{-1}) = (x^{\frac{1}{2}} \cdot yx^{\frac{1}{2}})(x^{-\frac{1}{2}} \cdot y^{-1}x^{-\frac{1}{2}})$. Using the Bol identity we see that the last expression is equal to 1.

It remains to prove that the loop $L(\frac{1}{2}, 1)$ is a left A-loop. We have seen already that the mappings $\lambda^*_{(y)}(u)$: $L(\frac{1}{2}, y) \to L(\frac{1}{2}, u)$ are isomorphisms of loops. Hence the mappings $\lambda^*_{(1)}(y)\lambda^*_{(y)}(x *_1 y)[\lambda^*_{(1)}(x *_1 y)]^{-1}$ $(x, y \in L)$ are automorphisms of $L(\frac{1}{2}, 1)$. Now we express these mappings in terms of left translations of the loop $L(\frac{1}{2}, 1)$. Since

$$(x *_1 z)\lambda^*_{(1)}(y) = x\lambda^*_{(1)}(y) *_y z\lambda^*_{(1)}(y)$$

or equivalently

$$z\lambda^*_{(1)}(x)\lambda^*_{(1)}(y) = z\lambda^*_{(1)}(y)\lambda^*_{(y)}(x *_1 y)$$

we obtain

$$\lambda^*_{(y)}(x *_1 y) = \lambda^*_{(1)}(y)^{-1}\lambda^*_{(1)}(x)\lambda^*_{(1)}(y).$$

Hence the mappings $\lambda^*_{(1)}(y)\lambda^*_{(y)}(x *_1 y)\lambda^*_{(1)}(x *_1 y)^{-1} = \lambda^*_{(1)}(x)\lambda^*_{(1)}(y)\lambda^*_{(1)}(x *_1 y)^{-1}$ are automorphisms and the theorem is proved. ∎

A quasigroup $(Q, \circ)$ is called left distributive if it satisfies the following identity: $x \circ (y \circ z) = (x \circ y) \circ (x \circ z) (x, y, z \in Q)$.

COROLLARY 2.8: *Let $(Q,+)$ be a left distributive quasigroup satisfying the identity $x+(x+y)=y$ for all $x,y\in Q$. Then the loop $Q^*(a)$ defined on the set Q by the multiplication*

$$x *_a y = xR^+(a)^{-1} + yL^+(a)$$

with unit element a has the following properties:

(i) *It is a Bol loop with automorphic inverse property.*

(ii) *It is a left A-loop.*

All loops $Q^*(a)$ ($a \in Q$) are isomorphic and their cores $(Q,\oplus_a)$ are isomorphic to $(Q,+)$ under the mapping $R^+(a)^{-1}$.

Proof: Since all loops isotopic to a Bol loop are Bol, $Q^*(a)$ is a Bol loop (cf. Theorem 9.11 in [3], p. 171). The inverse element of $x\in Q^*(a)$ is $a+x$. Indeed, $x *_a (a+x) = xR^+(a)^{-1} + x = xR^+(a)^{-1} + (xR^+(a)^{-1}+a) = a$ since $u+(u+a)=a$. Now, the automorphic inverse property follows from the left distributivity $a+(x+y)=(a+x)+(a+y)$.

Property (ii) and the fact that all loops $Q^*(a)$ are isomorphic follow from the proof of the previous theorem.

The operation $\oplus_a$ of the core of $Q^*(a)$ is given by

$$\begin{aligned} x\oplus_a y &= x *_a (y^{-1} *_a x) = x *_a [(a+y)R^+(a)^{-1} + xL^+(a)] \\ &= xR^+(a)^{-1} + [(a+y)R^+(a)^{-1} + xL^+(a)]L^+(a) \\ &= xR^+(a)^{-1} + [(a+y)R^+(a)^{-1}L^+(a) + x] \\ &= [xR^+(a)^{-1} + (a+y)R^+(a)^{-1}L^+(a)] + [xR^+(a)^{-1} + (xR^+(a)^{-1}+a)] \\ &= [xR^+(a)^{-1} + (a+y)R^+(a)^{-1}L^+(a)]R^+(a). \end{aligned}$$

Since $a+y=(a+y)R^+(a)^{-1}R^+(a) = (a+y)R^+(a)^{-1}+a$ we have $y = (a+y)R^+(a)^{-1}L^+(a)+a$ or $(a+y)R^+(a)^{-1}L^+(a) = yR^+(a)^{-1}$. Hence

$$(x\oplus_a y)R^+(a)^{-1} = xR^+(a)^{-1} + yR^+(a)^{-1}\,. \qquad \blacksquare$$

Now we give some generalizations of Theorems 10.20–10.22 in [2], pp. 52–55.

THEOREM 2.9: *Let L be a differentiable Bol loop (of class C^∞). Then the following groups are Lie transformation groups:*

(i) *the group $\mathring{A}$ of continuous automorphisms of the core $(L,+)$ of the loop L;*

(ii) *the group of continuous semiautomorphisms of L;*

(iii) *the group of continuous right pseudoautomorphisms of L;*

(iv) *the group of continuous automorphisms of* L.

Proof: For a differentiable Bol loop L we consider the associated 3-net on $L \times L$. We have proved in Proposition 1.5 that putting $\mu(x,y) = y\sigma_x^L$ the line $L \times \{1\}$ becomes a symmetric space. Using the original loop multiplication on L the multiplication μ can be expressed as $\mu(x,y) = x \cdot y^{-1}x = x + y$, where the operation "+" is core-addition. Now, (i) follows from Theorem 2.8, p. 88 and 1.7, p. 129 in [18]. The group of semiautomorphisms is the stabilizer subgroup of the point 1 in the group $\mathring{A}$, and hence it is a Lie group (cf. Theorem 2.5 (i)). A right pseudoautomorphism of the loop L is a permutation γ: $L \to L$ such that there exists an element $c \in L$ satisfying the identity $x^\gamma \cdot (y^\gamma c) = (xy)^\gamma \cdot c$ for all $x, y \in L$. The right pseudoautomorphisms of a Bol loop are semiautomorphisms since

$$(x \cdot yx)^\gamma \cdot c = x^\gamma((yx)^\gamma c) = x^\gamma(y^\gamma \cdot x^\gamma c) = (x^\gamma \cdot y^\gamma x^\gamma) \cdot c$$

and

$$1^\gamma \cdot c = (1 \cdot 1)^\gamma c = 1^\gamma \cdot 1^\gamma c \ ,$$

thus $1^\gamma = 1$. Hence the group of right pseudoautomorphisms is a closed subgroup of the group of semiautomorphisms and (iii) holds. The statement (iv) is an immediate consequence of (iii). ∎

The next statement shows that differentiable left distributive quasigroups have the same quality as the analytic ones.

THEOREM 2.10: *Let* $(L,+)$ *be a topological connected left distributive quasigroup on a manifold with differentiable multiplication and satisfying the identity* $x + (x + y) = y$ *for all* $x, y \in L$. *Then:*

(i) *The quasigroup* $(L,+)$ *is analytic.*

(ii) *The group (topologically) generated by the left translations of* $(L,+)$ *is a Lie group.*

(iii) *The loops* $L^*(a)$ *defined by the multiplications* $(x,y) \mapsto x *_a y = xR^+(a)^{-1} + yL^+(a)$ *are analytic.*

(iv) *The group* Γ^a *(topologically) generated by the left translations* $\overset{*}{\underset{(a)}{\lambda}}{}_x^{-1}$ *of* $L^*(a)$ *is a Lie transformation group for any* $a \in L$.

Proof: With respect to the multiplication "+" L is a symmetric space with the property that $x + y = y$ implies $y = x$ for all $x, y \in L$ (cf. p. 63 in [18]). From Corollary on p. 94 in [18] follows that the operation "+" is analytic. This means $(x,y) \mapsto yL^+(x)$ is analytic. The solution with respect to x of the equation $a + x = b$ depends analytically on a and b since $L^+(a) = [L^+(a)]^{-1}$.

The symmetric space L has a representation on the space of symmetric elements of a Lie group (cf. [18], p. 73). This means the following: there exists a connected Lie group G and an involutive automorphism $\sigma: G \to G$ such that L is realized on the connected component G_σ of $1 \in G$ in the set $\{x \in G,\ x^\sigma = x^{-1}\}$ with respect to the multiplication $x + y = xy^{-1}x$ for all $x, y \in G_\sigma$ (cf. [18], Theorem 4.6 and Proposition 4.4, pp. 182–185).

Since the equation $za^{-1}z = z + a = b$ has a unique solution in G_σ and 1 is contained in G_σ the mapping $z \mapsto z^2$ is bijective on G_σ. We denote the solution of $z + 1 = b$ by $z = b^{\frac{1}{2}}$. The solution of the equation $x + a = b$ can be expressed by $x = a^{\frac{1}{2}} + (a^{\frac{1}{2}} + b)^{\frac{1}{2}}$ since $x + a = b$ is equivalent to $(a^{\frac{1}{2}} + x) + (a^{\frac{1}{2}} + a) = (a^{\frac{1}{2}} + x) + 1 = a^{\frac{1}{2}} + b$ and hence $a^{\frac{1}{2}} + x = (a^{\frac{1}{2}} + b)^{\frac{1}{2}}$.

Now, we show that every element in G_σ is contained in precisely one 1-parameter subgroup of G_σ. Let x be an element of G_σ and $\overline{\langle x \rangle}$ be the monothetic subgroup of G generated by x. Since G_σ is closed in G the group $\overline{\langle x \rangle}$ is contained in G_σ. The group $\overline{\langle x \rangle}$ is isomorphic to $\mathbb{Z}$ or it is compact (cf. [8], p. 85). If $\overline{\langle x \rangle}$ is compact we denote by K a maximal compact subgroup of G containing $\overline{\langle x \rangle}$. Since the restriction of the exponential map from the Lie algebra $\mathfrak{g}$ of G to the Lie subalgebra $\mathfrak{k}$ of K is surjective onto K ([9], p. 150, Theorem 3.2) there exists a vector $X \in \mathfrak{k}$ such that $x = \exp X$. We have $x^\sigma = x^{-1} = \exp(-X)$. For the automorphism $\sigma_*: \mathfrak{g} \to \mathfrak{g}$ induced by σ we have $(tX)^{\sigma_*} = -tX$ for all $t \in \mathbb{R}$. But then G_σ would contain a 1-dimensional torus subgroup which contradicts the fact that the mapping $z \mapsto z^2$ is bijective. Hence for every $x \in G_\sigma$ we have $\overline{\langle x \rangle} \cong \mathbb{Z}$. Then the closure $\overline{\langle x \rangle}_2$ in G_σ of the group containing $\overline{\langle x \rangle}$ and with any element also its square root is isomorphic to $\mathbb{R}$. Since for $x, y \in G_\sigma$ and $\overline{\langle x \rangle}_2 \neq \overline{\langle y \rangle}_2$ one has $\overline{\langle x \rangle}_2 \cap \overline{\langle y \rangle}_2 = \{1\}$ and every element of G_σ different from 1 is contained in precisely one 1-parameter subgroup. It follows that the mapping $x \mapsto x^{\frac{1}{2}}: G_\sigma \to G_\sigma$ is analytic. Hence the mapping $(a, b) \mapsto b/a = bR^+(a)^{-1} = a^{\frac{1}{2}} + (a^{\frac{1}{2}} + b)^{\frac{1}{2}}: L \to L$ is analytic too.

The assertion (ii) follows from Theorem 2.9 (i) since the left translations of $(L, +)$ are automorphisms of $(L, +)$.

The assertion (iii) follows since the operations of $L^*(a)$ can be expressed by:

$$x *_a y = xR^+(a)^{-1} + yL^+(a),$$
$$y \underset{(a)}{\overset{*}{\lambda}}{}_x^{-1} = yL^+(xR^+(a)^{-1})L^+(a) = a + [xR^+(a)^{-1} + y],$$
$$x \underset{(a)}{\overset{*}{\rho}}{}_y^{-1} = xR^+(yL^+(a))^{-1}R^+(a) = xR^+(a + y)^{-1} + a.$$

The group Γ^a is a Lie group since it is the group of displacements (i.e. generated by $L^+(x)L^+(y)$, $x, y \in (L, +)$) of the symmetric space L^+. ∎

At the end of this section we show that for analytic Bol loops and analytic left distributive quasigroups the groups (topologically) generated by the right translations are usually not Lie groups.

Let H^2 be the hyperbolic plane. It can be considered as a symmetric space $(L, +)$ with respect to the operation $(x, y) \to x + y = y\sigma_x$, where σ_x denotes the reflection on the point x. The group generated by the left translations $L^+(x) = \sigma_x$, $x \in H^2$, is the connected component $J^0(H^2)$ of the isometry group $J(H^2)$ of H^2.

Now we consider the group Ψ generated by the right translations $R^+(x)$, $x \in H^2$. The action of the right translation $R^+(a)$ on H^2 can be described as follows: a is the only fixed point of $R^+(a)$, the invariant lines of $R^+(a)$ are the lines through a, the image $xR^+(a)$ of a point x lies on the half line emanating from a and containing x such that $(a, xR^+(a)) = 2d(a, x)$ where d is the distance function on H^2. From this description follows that $\lambda^{-1}R^+(a)\lambda = R^+(a^\lambda)$ for any isometry λ of H^2. Hence the group Ψ is normalized by the isometry group of H^2. The intersection $\Psi \cap J(H^2) = \{1\}$ since all elements of Ψ fix every end (i.e. in the Cayley model every point of the absolute conic) of the hyperbolic plane.

Let us assume that Ψ is a Lie group. Since the stabilizer Ψ_L of any line L in Ψ operates transitively on L the group Ψ is transitive on the points of H^2. If the connected component Ψ_L^0 of Ψ_L were not transitive on L then it would leave a point $x \in L$ fixed. Since Ψ_L^0 is a normal subgroup of Ψ_L the group Ψ_L^0 would fix the line L pointwise. But this is not possible since the topological dimension of Ψ_L/Ψ_L^0 is zero (cf. [7]). Hence Ψ_L^0 is transitive on L. If the connected component Ψ^0 of the group Ψ were not transitive on H^2 then Ψ^0 would have two different orbits B_1 and B_2. Let L be a line having points in common with both B_1 and B_2. Since the connected group Ψ_L^0 operates transitively on L we would have a contradiction and the group Ψ^0 is transitive on H^2. Moreover one has $\dim \Psi^0 \geq 2$. If $\dim \Psi^0 = 2$ then Ψ^0 operates sharply transitively on H^2 and one has $\Psi^0 \cong \mathbb{R}^2$ or $\Psi^0 \cong \{x \mapsto ax + b;\ a > 0, b \in \mathbb{R}\}$. Clearly $\Psi = \Psi^0\Psi_x$ with $\Psi^0 \cap \Psi_x = \{1\}$ where $\Psi_x \neq \{1\}$ is the stabilizer of a point x in Ψ. The element $R^+(x) \in \Psi_x$ induces an automorphism $\neq 1$ on Ψ^0 so that this automorphism leaves infinitely many 1-parameter subgroups of Ψ^0 invariant, namely any stabilizer Ψ_G^0 where G is a line through x. Hence Ψ^0 cannot be the non-abelian 2-dimensional Lie group and one has $\Psi^0 \cong \mathbb{R}^2$.

Let L be a line and $w \in \Psi^0$ with $w(L) \neq L$. Since the closure of $w(L)$ contains the two ends of L the orbit $w(L)$ is not a line of H^2. Let u be a point of $w(L)$ and

$1 \neq \tau \in \Psi^0_L$. Let L' be the line joining u and $\tau(u)$. Then in the stabilizer $\Psi^0_{L'}$ there exists an element ϕ with $\phi(u) = \tau(u)$. Since Ψ^0 operates sharply transitively on H^2 one has $\tau = \phi$ and $\Psi^0_{L'} \cap \Psi^0_L \neq \{1\}$. But this is a contradiction since in the Lie group $\mathbb{R}^2$ two different 1-dimensional subgroups intersect each other trivially. Hence $\dim \Psi \geq 3$. If Ψ were a Lie group then the semidirect product of the groups Ψ and $J^0(H^2)$ in the group of diffeomorphisms of H^2 would be an at least 6-dimensional Lie group containing the proper subgroup $J^0(H^2) \cong \mathrm{PSL}_2(\mathbb{R})$. The list of Lie transformation groups acting on the plane given in [23], §10, does not contain such a group. Consequently Ψ cannot be a Lie group.

Now we consider the translation groups of the loop $L^*(1)$, where 1 is a given point of H^2. Since $x *_1 y = xR^+(1)^{-1} + yL^+(1)$ the group Φ^* of left translations is generated by the products $\sigma_1 \sigma_{xR^+(1)^{-1}}$ of reflections ($x \in H^2$); this group can be generated also by the products $\sigma_u \sigma_v$, $(u, v \in H^2)$ and one has that Φ^* is isomorphic to $\mathrm{PSL}_2(\mathbb{R})$.

The group Ψ^* generated by the right translations $R^+(1)^{-1} R^+(yL^+(1))$ of $L^*(1)$ is connected and contained in the connected component Ψ^0 of the group Ψ which is topologically generated by the elements $R^+(x)$, $x \in H^2$. Since Ψ^* has index at most 2 in Ψ^0 also Ψ^* cannot be a Lie group.

§3. Hausdorff–Campbell formulae for geodesic and for left alternative loops

Let L be a C^r-manifold with $r \in \mathbb{N} \cup \{\infty, \omega\}$. Let $\cdot$, $\backslash$ and $/$ be C^r-mappings from open domains of $L \times L$ to L such that

$$(x/y) \cdot y = x,\ y \cdot (y \backslash x) = x,\ (xy)/y = x \text{ and } y \backslash (yx) = x$$

if the left side of the identity is defined.

Then L is called a local quasigroup of the class C^r. A local quasigroup L of class C^r is called a local loop of the class C^r if moreover the following condition is satisfied: There is an element e of L such that $xe = ex = x/e = e\backslash x = x$ for all $x \in L$.

Let L be a differentiable manifold equipped with an affine connection ∇. We can associate with a point $e \in L$ and with ∇ a local loop $\{L, \nabla, e\}$, called a geodesic loop with respect to ∇, which is defined in a normal neighbourhood U of e by the formula

$$x \cdot y = \exp_y \circ \tau_{e,y} \circ \exp_e^{-1}(x) \ ,$$

where $\tau_{e,y}$ denotes the parallel translation $T_e L \to T_y L$ along the unique geodesic segment between e and y (cf. [13], p. 160 and [21], p. 369).

The differentiable Bol loops have local representations as geodesic loops, but the class of differentiable loops which can be locally represented as geodesic loops is much larger. Namely, one has the following result due to [21], pp. 370 and 406.

PROPOSITION 3.1: *A differentiable local loop L which is simply covered by 1-parameter subgroups has a local representation as a geodesic loop with respect to a unique affine connection ∇ on L with vanishing curvature if and only if in L one has*

(i) $$x^t \cdot x^s y = x^{t+s} y$$

for all $x, y \in L$ and $t, s \in \mathbb{R}$ as far as the multiplication is defined. If L satisfies (i) then the connection ∇ is given by the parallel vector fields $(T_e\rho_y)v$, $v \in T_eL$, where $T_e\rho_y \colon T_eL \to T_yL$ is the tangent map of the right translation $\rho_y \colon x \mapsto xy$.

Miheev and Sabinin [20], p. 11 and p. 45 (cf. also [21], p. 406, Remark XII.7.1), proved that analytical local loops satisfying the identity

(ii) $$x \cdot xy = x^2 \cdot y$$

are power associative and fulfil identity (i). Hence they are geodesic loops with respect to an affine connection ∇ with zero curvature.

This motivates us to study left alternative loops, i.e. loops satisfying the identity (ii) in the category of real analytic loops, and to use their local representations as geodesic loops.

First we remark that in geodesic loops L the 1-parameter subgroups determine the local exponential mapping exp: $T_eL \to L$ which coincides with the exponential mapping at $e \in L$ of the linear connection ∇ of the geodesic loop L, since the 1-parameter subgroups of L are the geodesic lines of ∇ through $e \in L$. Hence for any geodesic loop L the canonical coordinate system (determined by the 1-parameter subgroups) is a normal coordinate system of the connection ∇. Since the connection ∇ has vanishing curvature the parallel translation with respect to ∇ is locally path-independent and can be given by independent parallel vectorfields $A_1(x), \dots, A_n(x)$ in the coordinate system $U \subset \mathbb{R}^n$, where $A_i(0) = e_i$ gives the canonical basis of $\mathbb{R}^n$. Since the parallel vectorfields determine the parallel translation $\tau_{e,y}$ by $e_i \mapsto A_i(y)$ $(i = 1, \dots, n)$ the map $\sum_{i=1}^n \xi^i e_i \mapsto \sum_{i=1}^n A_i(y)\xi^i$ gives the tangent map $T_e\rho_y$ of the right translation $\rho_y \colon x \mapsto xy$. In the normal coordinate system the geodesic lines through 0 are the euclidean lines. We want to find the power series expansion of the geodesic lines $y(t)$ in a neighbourhood of a point $y_0 \in U$. The differential equations for the coordinate functions $y^i(t)$

of the geodesic line $y(t)$ with initial values $y(0) = y_0$ and $\dot{y}(0) = \sum_j A_j(y_0)\xi^j$ are given by

$$\dot{y}^i(t) = \sum_j A^i_j(y(t))\xi^j, \tag{A}$$

where A^i_j denote the coordinate functions of the vectorfields A_j and ξ^j are the coordinates of $\dot{y}(0)$ with respect to the basis $A_j(y_0)$. Since any line $(\xi^1,\dots,\xi^n)t$ is a geodesic line, equation (A) implies that the coordinate functions $A^i_j(x)$ of the vectorfields A_j have to satisfy the identities

$$\xi^i = \sum_j A^i_j(\xi^1,\dots,\xi^n)\xi^j \ .$$

Their power series expansion at $0 = (0,\dots,0)$ gives

$$\begin{aligned}\xi^i &= \sum_j^n \Big[A^i_j(0) + \sum_{r=1}^{\infty}\frac{1}{r!}\sum_{j_1,\dots,j_r}^{n}\frac{\partial^r A^i_j}{\partial x^{j_1}\cdots\partial x^{j_r}}(0)\xi^{j_1}\dots\xi^{j_r}\Big]\xi^j\\ &\equiv \xi^i + \sum_{r=1}^{\infty}\frac{1}{r!}\sum_{j_1,\dots,j_{r+1}=1}^{n}\frac{\partial^r A^i_{j_{r+1}}}{\partial x^{j_1}\dots\partial x^{j_r}}(0)\xi^{j_1}\dots\xi^{j_r}\xi^{j_{r+1}}\end{aligned}$$

since $A^i_j(0) = \delta^i_j$. Consequently the coefficients of this expansion satisfy

$$\sum_{\sigma\in\mathbb{Z}_{r+1}}\frac{\partial^r A^i_{j_{(r+1)\sigma}}}{\partial x^{j_{1\sigma}}\cdots\partial x^{j_{r\sigma}}}(0) = 0, \tag{B}$$

where $\mathbb{Z}_{r+1}$ is the cyclic permutation group of order $r+1$.

Now we consider the power series expansion of the geodesic line $y(t)$ at $y(0) = y_0$ and $\dot{y}(0) = \sum_{i=1}^n\Big(\sum_{m=1}^n A^i_m(y_0)\xi^m\Big)e_i$. Since the geodesic line $y(t)$ satisfies $\nabla_{\dot{y}}\dot{y} = 0$ we have

$$\dot{y}(t) = \sum_{i=1}^n\Big(\sum_{m=1}^n A^i_m\big(y(t)\big)\xi^m\Big)e_i \quad \text{for all } t.$$

Consequently the power series of the geodesic line $\sum_{i=1}^n y^i(t)e_i$ at $\sum_{i=1}^n y^i_0 e_i = \sum_{i=1}^n y^i(0)e_i$ has the form

$$y^i(t) = y^i_0 + \sum_{q=1}^{\infty}\frac{t^q}{q!}\sum_{j_1,\dots,j_q=1}^{n} G^i_{j_1\cdots j_q}(y_0)\xi^{j_1}\cdots\xi^{j_q}, \tag{C}$$

where

(C¹)
$$G^i_j(y_0) = A^i_j(y_0) \quad \text{and}$$
$$G^i_{j_1\cdots j_{q-1}j_q}(y_0) = \frac{1}{q}\sum_{\sigma\in\mathbb{Z}_q}\sum_{m=1}^{n}\frac{\partial G^i_{j_{1\sigma}\cdots j_{(q-1)\sigma}}(y_0)}{\partial x^m}G^m_{j_{q\sigma}}(y_0).$$

The exponential map $\exp_0$ is given in the normal coordinate system by the identity mapping, the parallel translation τ_{0,y_0} is determined by the vectorfields $A^i_j(y_0)$ and the images of $\exp_{y_0}$ are described by the geodesic lines through y_0. Hence we can write for the loop multiplication

(D)
$$\begin{cases} t(\xi^1\cdots\xi^n)\cdot y_0 = \exp_{y_0}\cdot\tau_{0,y_0}[t(\xi^1\cdots\xi^n)] = y(t) \\ = \Big(y_0^i + \sum_{q=1}^{\infty}\frac{t^q}{q!}\sum_{j_1,\dots,j_q=1}^{n} G^i_{j_1\ \ j_q}(y_0)\xi^{j_1}\cdots\xi^{j_q}\Big)_{i=1}^{n}. \end{cases}$$

The desired expansion of loop multiplication will be obtained by power series extension of the functions $G^i_{j_1\cdots j_q}(y)$ varying y:

$$G^i_{j_1\cdots j_q}(y) = G^i_{j_1\cdots j_q}(0) + \sum_{p=1}^{\infty}\frac{1}{p!}\sum_{h_1,\dots,h_p=1}^{n}\frac{\partial^p G^i_{j_1\cdots j_q}}{\partial x^{h_1}\cdots\partial x^{h_p}}(0)y^{h_1}\cdots y^{h_p}\ .$$

If we put $y_0 = 0$, then from (D) it follows that

$$\begin{aligned} t(\xi^1,\dots,\xi^n) &= t(\xi^1,\dots,\xi^n)\cdot(0,\dots,0) \\ &= \Big(\sum_{q=1}^{\infty}\frac{t^q}{q!}\sum_{j_1,\dots,j_q=1}^{n} G^i_{j_1\cdots j_q}(0)\xi^{j_1}\cdots\xi^{j_q}\Big)_{i=1}^{n}. \end{aligned}$$

Since $G^i_j(0) = A^i_j(0) = \delta^i_j$ where (δ^i_j) is the identity matrix, the last relation can be written in the form

$$t\xi^i = t\xi^i + \sum_{q=2}^{\infty}\frac{t^q}{q!}\sum_{j_1,\dots,j_q=1}^{n} G^i_{j_1\cdots j_q}(0)\xi^{j_1}\cdots\xi^{j_q}$$

and we have $G^i_{j_1\cdots j_q}(0) = 0$ for all $j_1,\dots,j_q$ with $q > 1$.

For $p, q \geq 1$ we put

(E)
$$a^i_{j_1\cdots j_q;h_1\cdots h_p} := \frac{1}{p!q!}\frac{\partial^p G^i_{j_1\cdots j_q}}{\partial x^{h_1}\cdots\partial x^{h_p}}(0).$$

This definition implies the relation $a_{j_{1\sigma}\cdots j_{q\sigma},h_{1\rho}\cdots h_{p\rho}} = a_{j_1\cdots j_q;h_1\cdots h_p}$ for any permutations σ and ρ in the symmetric groups on q and p elements, respectively. The identities (B) give the equivalent conditions

$$\text{(F)} \qquad \sum_{\sigma\in\mathbb{Z}_{r+1}} a^{i}_{k_{1\sigma};k_{2\sigma}\cdots k_{r\sigma}} = 0.$$

The Leibniz formula for several variables has the form

$$\frac{\partial^p}{\partial x^{h_1}\cdots\partial x^{h_p}}[f(x^1,\dots,x^n)g(x^1,\dots,x^n)] = \sum_{p'+p''=p} \frac{\partial^{p'} f(x^1,\dots,x^n)}{\partial x^{h_{1'}}\cdots\partial x^{h_{p'}}}\frac{\partial^{p''} g(x^1,\dots,x^n)}{\partial x^{h_{1''}}\cdots\partial x^{h_{p''}}},$$

where $(I',I'')\in\mathcal{P}(2,p)$, $I'=\{h_{1'},\dots,h_{p'}\}$, $I''=\{h_{1''},\dots,h_{p''}\}$ and $\mathcal{P}(2,p)$ denotes the partitions of the set $\{1,\dots,p\}$ into two subsets. Applying this Leibniz formula to the equation (C^1) with $q>1$ and using the relation (E) we obtain

$$\text{(G)}\begin{cases} a^{i}_{j_1\cdots j_q;h_1\cdots h_p} = \dfrac{1}{q\cdot p!\cdot q!}\displaystyle\sum_{\substack{p'+p''=p\\ m=1,\dots,n\\ \sigma\in\mathbb{Z}_q}} \frac{\partial^{p'+1}G^{i}_{j_{1\sigma}\cdots j_{(q-1)\sigma}}}{\partial x^{h_{1'}}\cdots\partial x^{h_{p'}}\partial x^{m}}(0)\frac{\partial^{p''}G^{m}_{j_{q\sigma}}}{\partial x^{h_{1''}}\cdots\partial x^{h_{p''}}}(0)\\ = \dfrac{1}{q\cdot p!q!}\displaystyle\sum_{\substack{p'+p''=p\\ m=1,\dots,n\\ \sigma\in\mathbb{Z}_q}} (q-1)!(p'+1)!p''!\,a^{i}_{j_{1\sigma}\cdots j_{(q-1)\sigma};h_{1'}\cdots h_{p'}m}\,a^{m}_{j_{q\sigma},h_{1''}\cdots h_{p''}}.\end{cases}$$

Using the formula (D) with $t=1$, $\xi_i=x_i$, $y_0=y$ and the power series expansion of $G^i_j(y)=A^i_j(y)$ as well as of $G^i_{j_1\cdots j_q}(y)$, $q>1$, we see that the local loop multiplication $x\cdot y$ can be written in the form

$$\begin{aligned}
&(x^1,\dots,x^n)\cdot(y^1,\dots,y^n)\\
&=\Big(y^i+\sum_{q=1}^{\infty}\frac{1}{q!}\sum_{j_1,\dots,j_q=1}^{n} G^{i}_{j_1\cdots j_q}(y)x^{j_1}\cdots x^{j_q}\Big)_{i=1}^{n}\\
&=\Big(\sum_{j=1}^{n}A^i_j(y)x^j+y^i+\sum_{q>1}^{\infty}\frac{1}{q!}\sum_{j_1,\dots,j_q=1}^{n} G^{i}_{j_1\cdots j_q}(y)x^{j_1}\cdots x^{j_q}\Big)_{i=1}^{n}\\
&=\sum_{j=1}^{n}\Big\{G^i_j(0)+\sum_{p=1}^{\infty}\frac{1}{p!}\sum_{h_1,\dots,h_p=1}^{n}\frac{\partial^p G^i_j}{\partial y^{h_1}\cdots\partial y^{h_p}}(0)y^{h_1}\cdots y^{h_p}\Big\}x^j+y^i\\
&\quad+\sum_{q>1}^{\infty}\frac{1}{q!}\sum_{j_1,\dots,j_q=1}^{n}\Big\{G^i_{j_1\cdots j_q}(0)\\
&\qquad\qquad+\sum_{p=1}^{\infty}\frac{1}{p!}\sum_{h_1,\dots,h_p=1}^{n}\frac{\partial^p G^i_{j_1\cdots j_q}}{\partial y^{h_1}\cdots\partial y^{h_p}}(0)y^{h_1}\cdots y^{h_p}\Big\}x^{j_1}\cdots x^{j_q}.
\end{aligned}$$

If we use (E) and the relations $G^i_{j_1\cdots j_q}(0) = 0$ $(q > 1)$ as well as $A^i_j(0) = G^i_j(0) = \delta^i_j$ we obtain finally

$$(H)\qquad (x^1,\ldots,x^n)\cdot(y^1,\ldots,y^n) = \Big(x^i + y^i + \sum_{p,q=1}^{\infty}\ \sum_{\substack{j_1,\ldots,j_q=1\\ h_1,\ldots,h_p=1}}^{n} a^i_{j_1\cdots j_q,h_1\cdots h_p} x^{j_1}\cdots x^{j_q} y^{h_1}\cdots y^{h_p}\Big)_{i=1}^{n}.$$

The coefficients

$$a^i_{j;h_1\cdots h_p} = \frac{\partial^p A^i_j}{\partial x^{h_1}\cdots\partial x^{h_p}}(0)$$

appearing in the relation (F) are the coefficients in the power series expansion of the vectorfields A_j at 0. The recursive formulae (G) show that they determine all the other coefficients of the power series expansion (H) of the loop multiplication.

The expansion (H) is a generalization of the classical Hausdorff–Campbell formula for local diassociative analytical loops (cf. [19]).

Conversely, if $\mathcal{S}^p(V^*)$ is the p-th symmetric power of the dual space of $V = \mathbb{R}^n$ and if we choose the tensors $a^i_{j;h_1\cdots h_p} \in V \otimes V^* \otimes \mathcal{S}^p(V^*)$ satisfying the linear equations (F) then they determine a formal power series such that the corresponding local loop multiplication (H) gives a geodesic loop associated with an affine connection ∇ having vanishing curvature.

As mentioned at the beginning of this section, the class of left alternative analytic local loops coincides with the class of geodesic loops with respect to an analytic affine connection with vanishing curvature. Hence we have the

PROPOSITION 3.2: *The power series expansion given in* (H) *holds for any left alternative analytic local loop L.*

A loop $(L,\cdot)$ satisfying $xy\cdot y = x\cdot y^2$ for all $x,y \in L$ is called a right alternative loop. Putting $x * y = y\cdot x$ we obtain a left alternative loop $(L,*)$ which is antiisomorphic to $(L,\cdot)$. Clearly the loop $(L,\cdot)$ is analytic and right alternative if and only if $(L,*)$ is analytic and left alternative. The loops $(L,\cdot)$ and $(L,*)$ have the same unit element and 1-parameter subgroups and therefore the same canonical coordinate system. Hence according to Proposition 3.2 the analytic right alternative loop $(L,\cdot)$ has the power series expansion given in (H) for $(L,*)$ if we interchange the role of the variables $(x^1,\ldots,x^n)$ and $(y^1,\ldots,y^n)$.

With these remarks we show now the

THEOREM 3.3: *Any analytic left and right alternative local loop L is diassociative and the power series* (H) *is the classical Hausdorff–Campbell formula with respect to a normal coordinate system.*

Proof: Since the loop $(L,\cdot)$ is left alternative the coefficients $a^i_{j_1\dots j_q;h_1\dots h_p}$ in the power series expansion (H) are determined by the coefficients $a^i_{j;h_1\dots h_p}$. The power series expansion (H) of the left alternative loop $(L,*)$ defined by $x*y = x\cdot y$ $(x,y\in L)$ has the coefficients $\overset{*}{a}{}^i_{j_1\dots j_q;h_1\dots h_p} = a^i_{h_1\dots h_p;j_1\dots j_q}$ which are determined by the coefficients $\overset{*}{a}{}^i_{j;h_1\dots h_p} = a^i_{h_1\dots h_p;j}$. It follows that all coefficients $a^i_{j_1\dots j_q,h_1\dots h_p}$ can be expressed as polynomials in the coefficients $a^i_{j;h} = -a^i_{h;j}$ (cf. (F)). Hence all coefficients of the expansion (H) are determined by the second order ones.

In the Akivis algebra $\mathfrak{L}$ of L the associator $\langle\cdot,\cdot,\cdot\rangle$ is defined in such a way (cf. [10], p. 239) that the identities $xx\cdot y = x\cdot xy$ and $xy\cdot y = x\cdot yy$ imply $\langle X,X,Y\rangle = \langle X,Y,Y\rangle = 0$ for all $X,Y\in\mathfrak{L}$. Hence the Akivis algebra $\mathfrak{L}$ is a binary Lie algebra (cf. [28], pp. 27–28 and [19]). The power series given by the classical Hausdorff–Campbell formula defines a local analytic diassociative loop S whose Akivis algebra is $\mathfrak{L}$. Since the canonical power series for the left and right alternative loops S and L coincide up to terms of order two, and these coefficients determine in both cases all other coefficients of the power series expansion we have $L = S$. ∎

Now we consider the power series expansion for geodesic analytic local loops with respect to a connection ∇ with vanishing curvature $R\equiv 0$ such that also the symmetric connection $\widetilde{\nabla}$ with the same geodesic lines and vanishing torsion $\widetilde{T}\equiv 0$ has the curvature tensor $\widetilde{R}\equiv 0$. This means that all geodesic lines of ∇ (and of $\widetilde{\nabla}$) in a normal neighbourhood of $e = 0\in V$ are euclidean lines. In this case in the formula (C) all $G^i_{j_1\dots j_q}(y)\equiv 0$ for $q>1$. Using the formula (C^1) we obtain the relation

$$\text{(I)}\qquad \sum_m\left(\frac{\partial A^i_j(y)}{\partial y^m}A^m_k(y) + \frac{\partial A^i_k(y)}{\partial y^m}A^m_j(y)\right)\equiv 0$$

which is necessary and sufficient for the geodesic lines to be euclidean lines. Since all $G^i_{j_1\dots j_q}\equiv 0$ for $q>1$ the formula (H) has the form

$$\text{(H}'\text{)}\quad (x^1\dots x^n)\cdot(y^1\dots y^n) = \left(x^i + y^i + \sum_{p=1}^{\infty}\sum_{h_1\dots h_p=1}^{n} a^i_{j;h_1\dots h_p}x^j y^{h_1}\dots y^{h_p}\right)_{i=1}^{n}$$

where the coefficients $a^i_{j;h_1\dots h_p}\in V\otimes V^*\otimes S^p(V^*)$ are given by

$$A^i_j(y) = \delta^i_j + \sum_{p=1}^{\infty}\sum_{h_1\dots h_p=1}^{n} a^i_{j;h_1\dots h_p}y^{h_1}\dots y^{h_p}$$

and satisfy the relation (I). We can see immediately that the multiplication function (H′) is linear in the first factor $(x^1 \dots x^n)$.

Any (local) geodesic loop L whose geodesic lines are euclidean lines has the property that all right translations $\rho_x\colon x \mapsto yx$ are affine transformations of the euclidean space. Hence we have the part (a) of the following

PROPOSITION 3.4:

(a) *The multiplication of a geodesic loop $(L, \circ)$ whose geodesic lines are euclidean lines can be represented as*

$$\xi \circ \eta = \xi + \eta + \xi B(\eta) = \eta + \xi\big(I + B(\eta)\big) ,$$

where ξ, η are row vectors of $\mathbb{R}^n$, $B(\eta)$ is a matrix function $\mathbb{R}^n \to \mathbb{R}^{n^2}$ such that $B(0) = 0$, I is the identity matrix and the right translations $\rho_\eta = I + B(\eta)$ form a locally sharply transitive set of a neighbourhood of $0 \in \mathbb{R}^n$.

(b) *The local loop $L = \mathbb{R}^n$ the multiplication of which is given by $\xi \circ \eta = \xi + \eta + \xi B(\eta)$ with $B(0) = 0$ is a geodesic loop with respect to an affine connection ∇ with vanishing curvature such that all geodesic lines of ∇ are euclidean lines if and only if the identity*

$$(*) \qquad \xi B(\eta) = \xi B\big(\xi + \eta + \xi B(\eta)\big)$$

holds for all ξ, η.

Proof: We must show the part (b) of the proposition. We have

$$(s\xi) \circ (t\xi) = (s + t)\xi + s\xi B(t\xi) = (s + t)\xi$$

for all $s, t \in \mathbb{R}$ if and only if $\xi B(\xi) = 0$, which is a necessary and sufficient condition that the curves $x(t) = t\xi$ are the 1-parameter subgroups of L and $0 \in \mathbb{R}^n$ is the identity of L.

Moreover, on the one hand

$$(s\xi)\circ[(t\xi)\circ\eta] = (s\xi)\circ[t\xi+\eta+t\xi B(\eta)] = (s+t)\xi+\eta+t\xi B(\eta)+s\xi B\big(t\xi+\eta+t\xi B(\eta)\big)$$

and on the other hand

$$[(s\xi) \circ (t\xi)] \circ \eta = (s + t)\xi \circ \eta = (s + t)\xi + \eta + (s + t)\xi B(\eta),$$

which are equal if and only if the identity $(*)$ holds. According to Proposition 3.1 in this case L is a geodesic loop with respect to the connection ∇ with vanishing curvature given by the parallel vector fields $[T_e\rho(\eta)]v = v\big(I + B(\eta)\big)$ ($v \in T_eL = \mathbb{R}^n$). The right multiplications of L are affine maps and the euclidean lines through 0 are geodesic lines. Hence all euclidean lines are geodesic lines with respect to ∇. ∎

Now we investigate the special case that the function $\eta \mapsto B(\eta)$ is linear. Then the function

$$(\xi, \eta) \mapsto \xi B(\eta) := [\xi, \eta]\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$$

is bilinear and anti-symmetric. Consequently the condition (∗) of Proposition 3.4(b) gives $[\xi, \eta] = \Big[\xi, \big[\xi + \eta + [\xi, \eta]\big]\Big]$ or equivalently $0 = \big[\xi, [\xi, \eta]\big]$ for all $\xi, \eta \in \mathbb{R}^n$.

THEOREM 3.5: *A mapping $(\xi, \eta) \mapsto \xi \circ \eta$: $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ defines a left and right alternative local analytic proper loop L such that the geodesic lines of the linear connections determined by the parallel vector fields $\xi \mapsto (T_0\lambda_\xi)v$ and $\eta \mapsto (T_0\rho_\eta)v$, $v \in \mathbb{R}^n$, respectively, are in both cases the euclidean lines if and only if $\xi \circ \eta = \xi + \eta + [\xi, \eta]$, where $(\xi, \eta) \mapsto [\xi, \eta]$ is the multiplication of a binary Lie algebra of nilpotency class 2 which is not a Lie algebra.*

Proof: Let L be a left and right alternative analytic local loop such that the euclidean lines are geodesic lines with respect to the two associated connections of L with vanishing curvature. Then Proposition 3.4 implies that the multiplication of L has the representations

$$\xi \circ \eta = \xi + \eta + \xi B^{(l)}(\eta) = \xi + \eta + \eta B^{(r)}(\xi)$$

with $\xi B^{(l)}(\eta) = \xi B^{(l)}\big(\xi + \eta + \xi B^{(l)}(\eta)\big)$ and $\eta B^{(r)}(\xi) = \eta B^{(r)}\big(\eta + \xi + \eta B^{(r)}(\xi)\big)$. Hence the multiplication of L has a representation $\xi \circ \eta = \xi + \eta + [\xi, \eta]$ with a bilinear mapping $(\xi, \eta) \mapsto [\xi, \eta]$ and $[\xi, \eta] = -[\eta, \xi]$. Moreover this bilinear mapping satisfies the identity $\big[\xi, [\xi, \eta]\big] = 0$ for all $\xi, \eta \in \mathbb{R}^n$. This means that $(\mathbb{R}^n, [.,.])$ is a binary Lie algebra of nilpotency class 2.

It follows from Theorem 3.3 that the analytic left and right alternative local loop L is diassociative and the expression $\xi \circ \eta = \xi + \eta + [\xi, \eta]$ is the classical Hausdorff–Campbell formula with respect to a normal coordinate system. Hence

$$[\xi, y] = \frac{1}{2} \lim_{t \to 0} t^{-2}\big((t\xi \circ t\eta)/(t\eta \circ t\xi)\big)$$

(cf. [10], p. 239) and the tangent algebra of L is a binary Lie algebra of nilpotency class 2. Clearly, L is associative if and only if the operation $(\xi, \eta) \mapsto [\xi, \eta]$ satisfies the Jacobi identity. There is a one-to-one correspondence between binary Lie algebras and analytic diassociative local loops. This completes not only the proof of the theorem, but also implies the

COROLLARY 3.6: *There is a one-to-one correspondence between the isomorphism classes of binary Lie algebras of nilpotency class 2 which are not Lie algebras, and the isomorphism classes of left and right alternative local analytic proper loops L such that the euclidean lines are geodesic lines with respect to the two associated connections of L with vanishing curvature.* ∎

An example for a Malcev algebra $\mathfrak{M}$ of dimension 5 and of nilpotency class 2 is given by Kuz'min [15], p. 694. There is a basis $e_1, \dots, e_5$ of $\mathfrak{M}$ such that $[e_1, e_2] = e_4$, $[e_3, e_4] = e_5$ and $[e_i, e_j] = 0$ otherwise. In this case the power expansion of the corresponding loop multiplication is finite and has the form

$$\begin{aligned}(\xi \circ \eta)^\alpha &= \xi^\alpha + \eta^\alpha \quad (\alpha = 1, 2, 3)\ ,\\ (\xi \circ \eta)^4 &= \xi^4 + \eta^4 + \xi^1\eta^2 - \xi^2\eta^1\ ,\\ (\xi \circ \eta)^5 &= \xi^5 + \eta^5 + \xi^3\eta^4 - \xi^4\eta^3\ .\end{aligned}$$

The next statement gives a characterization of the system of geodesic lines of the symmetric space $\mathcal{S}$ associated with a differentiable Bol loop L (cf. Proposition 1.5) in terms of 1-parameter subgroups of loops isotopic to the core of L.

PROPOSITION 3.7: *Let L be a differentiable Bol loop and let $\mathcal{S} = \{\sigma_x, x \in L\}$ be a set of reflections giving on L a structure of a symmetric space and defined by $\sigma_x(y) = x \cdot y^{-1}x$. Then the geodesic lines of the canonical connection $\widetilde{\nabla}$ of the symmetric space $(L, \mathcal{S})$ are precisely the 1-parameter subgroups of the loops $L_{(a)}$ having as multiplication $(x, y) \mapsto x \cdot a^{-1}y$.*

Proof: If 1 is the identity of L then $\sigma_a \circ \sigma_1 \colon x \mapsto a \cdot xa$. Moreover the 1-parameter subgroups of $L = L_{(1)}$ are geodesic lines of $\widetilde{\nabla}$ since the reflection $\sigma_1 \colon y \mapsto y^{-1}$ is a geodesic symmetry. If $x(t)$ is a 1-parameter subgroup of $L = L_{(1)}$ then $\sigma_a \circ \sigma_1\big(x(t)\big) = a \cdot x(t)a$ is a geodesic line with initial point $a \cdot (1 \cdot a) = a^2$, since $\sigma_a \circ \sigma_1$ is an affine map of $\widetilde{\nabla}$. Moreover, using the Bol identity and the multiplication of the loop $L_{(a^2)}$, we have

$$\big(a \cdot x(t)a\big) \cdot \big[a^{-2}\big(a \cdot x(s)a\big)\big] = a \cdot x(t)\big[a \cdot a^{-2}\big(a \cdot x(s)a\big)\big] = a \cdot x(t+s)a,$$

which means that it is a 1-parameter subgroup of $L_{(a^2)}$. The connected component Σ of the isometry group of $(L, \mathcal{S})$ is generated by the maps $\sigma_a\sigma_1$ $(a \in L)$ and the 1-parameter subgroups as well as the geodesic lines through a point p

simply cover a normal neighbourhood of p. Since the stabilizers of points p in Σ preserve the set of 1-parameter subgroups of $L_{(p)}$ the assertion follows.

Motivated by the previous proposition we want to characterize in terms of the power expansion and of the parallel vectorfields $A_{(1)} \dots A_{(n)}$ the class of differentiable local Bol loops L such that their local core is isotopic to a local abelian group. Such loops we call *flat* local Bol loops.

Let L be a flat local Bol loop such that for any $x \in L$ there is precisely one $y = x^{\frac{1}{2}}$ with $y^2 = x$. By Theorem 2.7 the local core $(L, +)$ of L is isotopic to the local loop $L(\frac{1}{2}, 1)$, the multiplication of which is given by $(x, y) \mapsto x^{\frac{1}{2}} \cdot yx^{\frac{1}{2}}$. A differentiable local Bol loop (Theorem 2.7) $L(\frac{1}{2}, 1)$ is a commutative local Lie group if and only if it is commutative, since commutative differentiable local Moufang loops are local groups. Hence the class of flat local Bol loops L is characterized by the identity $u \cdot v^2 u = v \cdot u^2 v$.

For any loop of this class the left translations of the core considered as the reflections of the corresponding local symmetric space (cf. proof of Proposition 2.3) generate an isometry group, the connected component of which is the euclidean translation group of the manifold L. Hence the canonical connection $\tilde{\nabla}$ of this flat symmetric space has vanishing curvature and vanishing torsion and all geodesic lines of $\tilde{\nabla}$ are euclidean lines in a normal neighbourhood of the unit of L.

A differentiable local Bol loop L is flat if and only if the ternary operation (X, Y, Z) of its Bol algebra $\mathfrak{L}$ satisfies $(X, Y, Y) = 0$ for all $X, Y \in \mathfrak{L}$ (cf. [21], Remark XII.8.14). The group topologically generated by the left translations of L is a Lie transformation group G (cf. Theorem 1.9.(iii) and [21], p. 424). If we denote by $\mathfrak{g}$ the Lie algebra of G and by $\mathfrak{M}$ the tangent space of the submanifold $\{\lambda_x, x \in L\}$ at $1 \in G$ (cf. [26], Proposition 2.2) then identifying $\mathfrak{M}$ with $\mathfrak{L}$ one has $(X, Y, Z) = \big[[X, Y], Z\big]$ (cf. [21], Proposition XII.8.25). Using [24], Theorem 2, we obtain that a differentiable local Bol loop L is flat if and only if $(X, Y, Z) = \big[[X, Y], Z\big] = 0$ for all $X, Y, Z \in \mathfrak{M}$. Since $\mathfrak{M}$ generates the Lie algebra $\mathfrak{g}$ it follows from [21], p. 422 and p. 424, that $\mathfrak{g}$ is a nilpotent Lie algebra of class 2 and of dimension $\leq n + \binom{n}{2}$ where $n = \dim L$. Then a differentiable local Bol loop L of dimension n is flat if and only if the group G is nilpotent of class 2 and of dimension $\leq n + \binom{n}{2}$.

Hence we have the following

THEOREM 3.8: *A differentiable local Bol loop L of dimension n is flat if and only if one of the following conditions is satisfied:*

(i) $x \cdot (y^2 \cdot x) = y \cdot (x^2 \cdot y)$ *for all* $x, y \in L$ *as far as defined.*

(ii) *The group (topologically) generated by the left translations of* L *is a Lie group of nilpotency class 2 and of dimension* $\leq n + \binom{n}{2}$.

Let $(L, \circ)$ be a differentiable flat local Bol loop and $(G, \cdot)$ the Lie group (topologically) generated by the left translations of L. We denote by $\mathfrak{g}$ the Lie algebra of G and by $\mathfrak{M}$ the tangent space of the submanifold $\{\lambda_x,\ x \in L\}$ at $1 \in G$.

We identify the group G of nilpotency class 2 with its Lie algebra $\mathfrak{g}$ via the exponential mapping. Then the flat local Bol loop L can be identified with an open neighbourhood of 0 in $\mathfrak{M}$. Let be $X_1, X_2 \in L \subset \mathfrak{M}$. We want to calculate the power series expansion $(tX_1) \circ (sX_2)$ $(t, s \in \mathbb{R})$. If $[X_1, X_2] = 0$ then the power expansion is trivial. If X_1 and X_2 do not commute in the Lie algebra $\mathfrak{g}$ then they generate a 3-dimensional nilpotent Lie subalgebra $\mathfrak{a}$ of $\mathfrak{g}$. The vectors $e_1 = X_1$, $e_2 = X_2$ and $e_3 = [X_1, X_2]$ form a basis of the Lie algebra $\mathfrak{a}$ such that the multiplication in the corresponding subgroup A of G is given by the Hausdorff–Campbell formula:

$$(u_1, v_1, z_1) \cdot (u_2, v_2, z_2) = \big(u_1 + u_2, v_1 + v_2, z_1 + z_2 + \tfrac{1}{2}(u_1 v_2 - u_2 v_1)\big).$$

Clearly te_1 and se_2 $(t, s \in \mathbb{R})$ generate a local subloop D of L which is an open subset of $\mathbb{R}e_1 + \mathbb{R}e_2 \subset \mathfrak{M}$. Let $\mathfrak{k}$ be the Lie algebra of the stabilizer K of $1 \in D$ in A. Since $\mathfrak{k} \cap \mathbb{R}e_3 = \{0\}$, the Lie algebra $\mathfrak{k}$ has the form $\mathfrak{k} = \mathbb{R}(\alpha e_1 - \beta e_2 + e_3)$ where $\alpha^2 + \beta^2 \neq 0$. The loop D can be represented on an open neighbourhood of $(0,0)$ of $\mathbb{R}e_1 + \mathbb{R}e_2$ by the multiplication

$$(x_1, x_2) \circ (y_1, y_2) = (z_1, z_2)$$

where

$$(x_1, x_2, 0) \cdot (y_1, y_2, 0) \cdot \mathfrak{k} = (z_1, z_2, 0) \cdot \mathfrak{k}.$$

From this we obtain the following system of equations:

$$\big(x_1 + y_1, x_2 + y_2, \tfrac{1}{2}(x_1 y_2 - x_2 y_1)\big) \cdot (t\alpha, -t\beta, t) = (z_1, z_2, 0)$$

for $t, z_1, z_2 \in \mathbb{R}$. This yields:

$$\begin{aligned} z_1 &= x_1 + y_1 + t\alpha\ , \\ z_2 &= x_2 + y_2 - t\beta\ , \\ 0 &= \tfrac{1}{2}(x_1 y_2 - x_2 y_1) + t - \tfrac{1}{2}[(x_1 + y_1)t\beta + (x_2 + y_2)t\alpha] \end{aligned}$$

or equivalently

$$(+)\qquad \begin{aligned} z_1 &= x_1 + y_1 + \frac{\alpha(x_2y_1 - x_1y_2)}{2 - \beta(x_1+y_1) - \alpha(x_2+y_2)},\\ z_2 &= x_2 + y_2 - \frac{\beta(x_2y_1 - x_1y_2)}{2 - \beta(x_1+y_1) - \alpha(x_2+y_2)}. \end{aligned}$$

Putting $x_1 = t$, $x_2 = 0$, $y_1 = 0$, $y_2 = s$ and using the power expansion

$$\frac{1}{2-w} = \sum_{i=0}^{\infty} \frac{w^i}{2^{i+1}}$$

we obtain the formula

$$(++)\qquad \begin{aligned} (tX_1)\circ(sX_2) &= \left[t - \alpha ts \sum_{i=0}^{\infty} \frac{(\beta t + \alpha s)^i}{2^{i+1}}\right] X_1 + \left[s + \beta ts \sum_{i=0}^{\infty} \frac{(\beta t + \alpha s)^i}{2^{i+1}}\right] X_2\\ &= tX_1 + sX_2\\ &\quad + (-\alpha X_1 + \beta X_2)ts \sum_{k,l=0}^{\infty} \frac{1}{2^{k+l+1}} \binom{k+l}{k} (\beta t)^k (\alpha s)^l. \end{aligned}$$

The coordinate functions $A_i^1(y_1, y_2)$ and $A_i^2(y_1, y_2)$ of the vector fields

$$A_i(y_1, y_2) = \big(T_{(0,0)}\rho(y_1, y_2)\big)X_i \quad (i = 1, 2),$$

where $\rho(y_1, y_2)$: $(x_1, x_2) \mapsto (x_1, x_2) \circ (y_1, y_2)$, can be calculated from the multiplication formula (+) by derivation:

$$A_1^1 = \left.\frac{\partial z_1}{\partial x_1}\right|_{(x_1,x_2)=(0,0)}, \quad A_2^1 = \left.\frac{\partial z_1}{\partial x_2}\right|_{(x_1,x_2)=(0,0)},$$
$$A_1^2 = \left.\frac{\partial z_2}{\partial x_1}\right|_{(x_1,x_2)=(0,0)} \quad \text{and} \quad A_2^2 = \left.\frac{\partial z_2}{\partial x_2}\right|_{(x_1,x_2)=(0,0)}.$$

It follows that

$$(+++)\qquad \begin{aligned} A_1^1(y_1, y_2) &= 1 - \frac{\alpha y_2}{2 - \beta y_1 - \alpha y_2}, & A_2^1(y_1, y_2) &= \frac{\alpha y_1}{2 - \beta y_1 - \alpha y_2},\\ A_1^2(y_1, y_2) &= \frac{\beta y_2}{2 - \beta y_1 - \alpha y_2}, & A_2^2(y_1, y_2) &= 1 - \frac{\beta y_1}{2 - \beta y_1 - \alpha y_2}. \end{aligned}$$

Collecting the results of the above investigations we obtain the following

THEOREM 3.9: *Let L be an analytic flat local Bol loop of dimension n. Denote by $\mathfrak{L}$ the Bol algebra of L, by G the nilpotent group (topologically) generated by the left translations of L and by H the stabilizer of $1 \in L$ in G. Then the following holds:*

(a) *If $\mathfrak{g}$ denotes the Lie algebra of G and $\mathfrak{H}$ the Lie algebra of H then $\mathfrak{L}$ is a complement of $\mathfrak{H}$ in $\mathfrak{g}$. The ternary operation in $\mathfrak{L}$ is trivial and the binary operation "$*$" in $\mathfrak{L}$ is given by $X_1 * X_2 = [X_1, X_2]_{\mathfrak{L}}$ for all $X_1, X_2 \in \mathfrak{L}$, where $[\ ,\ ]$ denotes the Lie product in $\mathfrak{g}$ and $[X_1, X_2]_{\mathfrak{L}}$ the projection of $[X_1, X_2]$ onto $\mathfrak{L}$ along $\mathfrak{H}$.*

(b) *Let X_1, X_2 be elements of $\mathfrak{L}$ with $[X_1, X_2] \neq 0$, let $A(X_1, X_2)$ be the 3-dimensional local nilpotent Lie group generated by $\exp tX_1$ and $\exp tX_2$ ($t \in \mathbb{R}$) and $\mathfrak{a}(X_1, X_2)$ its Lie algebra. Identifying $A(X_1, X_2)$ with $\mathfrak{a}(X_1, X_2)$ and $\exp tX_i$ with tX_i ($i = 1, 2$) the multiplication $(tX_1) \circ (tX_2)$ is given by the formula (++) , where the real numbers α and β are determined by*

$$\mathfrak{a}(X_1, X_2) \cap \mathfrak{H} = \mathbb{R}\big(\alpha X_1 - \beta X_2 + [X_1, X_2]\big).$$

(c) *The vector fields belonging to the tangent maps $T_{(0,0)}\rho(y_1, y_2)$: $T_{(0,0)}L \to T_{(y_1,y_2)}L$ with (y_1, y_2) in the local subloop of L generated by $\exp tX_1$ and $\exp sX_2$ ($t, s \in \mathbb{R}$) are given by the formula (+++) if we identify G with $\mathfrak{g}$ by the exponential map.*

Added in proof: A more direct proof of Theorem 3.3 was given independently by L. V. Sabinin in *On the diassociativity of smooth monoalternative loops*, Russian Mathematical Surveys **51** (1996), 747–749.

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